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EXACT SOLUTION FOR SZEKERES INHOMOGENEOUS COSMOLOGICAL MODELS IN STT OF GRAVITY [1]

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ABSTRACT

The Szekeres metric represents an exact inhomogeneous and anisotropic solution of the Einstein Equations and it has no Killing vector fields [Georg and Hellaby, Physical Review D 95, (2017)]. This means that this solutions presents a more general class of solutions. It is known to possess axial symmetry.

The inhomogeneous Szekeres cosmological models (ISCM) within the framework of scalar-tensor theory (STT) of gravity are obtained.

Inhomogeneous generalizations of the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmological models have gained interest in the astrophysical community and are more often employed to study cosmologicalphenomena.

In this paper, we first give the solutions of the STT field Equations. Andthen it's physical and geometrical properties of this metric and the reviews of recent developments in the field and shows the importance of an inhomogeneous framework in the analysis of cosmological observations.

Keywords: **ISCM, STT of gravity.**

1. INTRODUCTION

Einstein's general theory of relativity (GR) is a geometrical theory of space-time. The fundamental building block is a metric tensor field g_{ij} which is a tensor of rank two. Alternatives to GR are physical theories that attempt to describe the phenomenon of gravitation in competition to Einstein's theory of GR. In an alternative theory based on a metric tensor field along with another dynamical scalar field coupled to it is proposed by Brans and Dicke [4] accordingly called STT of gravity.

The theory formulated in [4] which is an STT of gravitation in which the tensor field alone is geometrized and the scalar field is aligned to the geometry. Sen and Dunn [12] have proposed a new STT of gravitation in which both the scalar and tensor fields have intrinsic geometrical significance.

The fields Equations in STT are given by,

$$G_{ij} + \frac{\omega(\phi)}{\phi^2}\left[\nabla_i\phi\nabla_j\phi - \frac{1}{2}g_{ij}\nabla_k\phi\nabla^k\phi\right] + \frac{1}{\phi}\left[\nabla_i\nabla_j\phi - g_{ij}\Box\phi\right] = \frac{8\pi}{\phi}T_{ij}, \qquad (1.1)$$

where G_{ij} is the Einstein tensor, $\omega(\phi)$ is some function of ϕ, ϕ is scalar field, ∇_i is the covariant derivative operator, $\Box = \nabla^k\nabla_k = g^{kl}\nabla_l\nabla_k$ is the d'Alembert operator for a scalar field. One can take the trace of Equation (1.1) overall space to g^{ij}, by using $g^{ij}G_{ij} = -R$, we obtain

$$(2\omega(\phi) + 3)\Box\phi = 8\pi T - \frac{d\omega}{d\phi}\nabla_k\phi\nabla^k\phi, \qquad (1.002)$$

where $T = g^{ij}T_{ij}$ is the trace of the stress-energy. Also, the matter satisfies the following conservation Equation

$$\nabla_i T^{ij} = 0. \qquad (1.3)$$

The conservation Equation gives above implies that the test-particles describes geodesics as in the case of GR.

[1]This paper was awarded, "Sardar Patel Research Award" in December 2018.

2. THE METRIC AND THE FIELD EQUATIONS

The Szekeres metric is given by [9]

$$ds^2 = dt^2 - e^{2A}(dx^2 + dy^2) - e^{2B}dz^2, \quad (2.1)$$

where the metric coefficients A and B are functions of all space-time co-ordinates i.e.,

$$A = A(x, y, z, t), B = B(x, y, z, t).$$

The computations of relevant tensors ; for metric (2.1) gives,

$$R = 2e^{-2A}\left(B_y^2 + A_{yy} + B_{yy} + B_x^2 + A_{xx} + B_{xx}\right) - 2e^{-2B}(3A_z^2 - 2A_zB_z + 2A_{zz}) + 2\left[3\dot{A}^2 + 2\dot{A}\dot{B} + \dot{B}^2 + 2\ddot{A} + \ddot{B}\right], \quad (2.2)$$

where an overhead dot denotes derivative with respect to time t and partial derivatives with respect to space variable are denoted by relevant subscripts e.g., $B_{xy} = \frac{\partial^2 B}{\partial x \partial y}$, etc.

The energy-momentum tensor for a perfect fluid is given by,

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij}, \quad (2.3)$$

where ρ is the proper energy density, p is the isotropic pressure and choosing a comoving observer we take, $u^i = (0,0,0,1)$ as 4-velocity of the fluid particles which satisfy the condition $u^i u_i = 1$. The average scale-factor $a(t)$, and spatial volume V are given by,

$$V = \sqrt{-g} = a^3 = e^{2A+B}. \quad (2.4)$$

In a cosmological setting, we define Hubble, deceleration, jerk, and snap parameters by,

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\sum_{i=1}^{3} H_i = \frac{1}{3}\left(2\dot{A} + \dot{B}\right), \quad (2.5)$$

$$q(t) = -\frac{\ddot{a}}{a}H^{-2} = -\left(1 + \frac{\dot{H}}{H^2}\right) = -\left(1 + \frac{2\ddot{A}+\ddot{B}}{3H^2}\right), \quad (2.6)$$

$$j(t) = \frac{\dddot{a}}{a}H^{-3} = q + 2q^2 - \frac{\dot{q}}{H} = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^2}, \quad (2.7)$$

$$s(t) = \frac{\ddot{a}}{a}H^{-4} = \frac{\ddot{a}a^3}{\dot{a}^4}. \quad (2.8)$$

In terms of the these parameters, we consider the following definitions

$$\dot{H} = -H^2(1 + q), \quad (2.9)$$
$$\ddot{H} = H^3(j + 3q + 2), \quad (2.10)$$
$$\dddot{H} = H^4(s - 2j - 5q - 3). \quad (2.11)$$

To realize how this term arises, consider the Taylor expansion of the scale factor, about the present time, t_0

$$a(t) = a_0 + \dot{a}_0(t - t_0) + \frac{1}{2}\ddot{a}_0(t - t_0)^2 + \ldots\ldots.., \quad (2.12)$$

where the sub-zeros indicate the terms are evaluated at the present. Using Equations of deceleration, jerk, and snap parameters are dimensionless, and we can write

$$a(t) = a_0\left[1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + \frac{1}{3!}j_0H_0^3(t - t_0)^3 + \frac{1}{4!}s_0H_0^4(t - t_0)^4 + O([t - t_0]^5)\right], \quad (2.13)$$

where H_0, q_0, j_0 and s_0, are the present time at $t = t_0$. In terms, the red-shift of Taylor expansion reads

$$\frac{1}{z+1} = H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + \frac{1}{3!}j_0H_0^3(t - t_0)^3 + \frac{1}{4!}s_0H_0^4(t - t_0)^4 + O([t - t_0]^5). \quad (2.14)$$

For small $H_0(t - t_0)$ this can be inverted to yield

$$t - t_0 = H_0^{-1}\left[z - (1 + \frac{q_0}{2})z^2 + \ldots..\right]. \quad (2.15)$$

If coordinates are chosen so cosmic time $t = 0$ denotes the time of the big bang (phase), then $t = 0$ is the age of the universe. The Hubble parameters (HPs) in the directions of x, y and z-axes are given by

$$H_1 = H_2 = \dot{A}, \qquad H_3 = \dot{B}. \quad (2.16)$$

The cosmological parameters such as the scalar expansion (θ), shear scalar (σ^2), shear parameter (Σ^2), and anisotropy parameter A_m are given by,

$$\theta = 3H = 2H_1 + H_3, \quad (2.17)$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - 3H^2\right) = \frac{1}{3}(H_1 - H_3)^2, \quad (2.18)$$

$$\Sigma^2 = \frac{\sigma^2}{3H^2} = \frac{1}{3}\left(\frac{H_1-H_3}{H}\right)^2, \quad (2.19)$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left[\frac{H_i-H}{H}\right]^2 = \frac{1}{3}\left(2\left[\frac{H_1-H}{H}\right]^2 + \left[\frac{H_3-H}{H}\right]^2\right), \quad (2.20)$$

where H_1, H_2, and $H_3$are given as in Equation (2.16).

3. EXACT SOLUTION FOR SZEKERES MODEL

Here we first develop some important cosmological parameters and EFEs for SMs and then find the exact solutions of EFEs. In fact, the differential Equation obtained in terms of the metric coefficient for$\phi = \phi(t)$is apparently not solvable.By using Equations (1.1), (2.1), and (2.3), we obtain a set of differential Equations for SMs,

$$\dot{H}_1 + \dot{H}_3 + H_1^2 + H_3^2 + H_1H_3 - e^{-2A}\left[B_y^2 + B_{yy} - A_yB_y + A_xB_x\right] + e^{-2B}[A_zB_z - A_z^2 - A_{zz}] + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + (H_1 + H_3)\frac{\dot{\phi}}{\phi} = -\frac{8\pi p}{\phi}, \quad (3.1)$$

$$\dot{H}_1 + \dot{H}_3 + H_1^2 + H_3^2 + H_1H_3 - e^{-2A}\left[B_x^2 + B_{xx} - A_xB_{yx} + A_yB_y\right] + e^{-2B}[A_zB_z - A_z^2 - A_{zz}] + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + (H_1 + H_3)\frac{\dot{\phi}}{\phi} = -\frac{8\pi p}{\phi}, \quad (3.2)$$

$$2\dot{H}_1 + 3H_1^2 - e^{-2A}\left[A_{yy} + A_{xx}\right] - e^{-2B}A_z^2 + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + (2H_1)\frac{\dot{\phi}}{\phi} = -\frac{8\pi p}{\phi}, \quad (3.3)$$

$$H_1^2 + 2H_1H_3 - e^{-2A}\left[A_{xx} + B_{xx} + B_y^2 + A_{yy} + B_{yy} + B_x^2\right] + e^{-2B}[2A_zB_z - 3A_z^2 - 2A_{zz}] - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + (2H_1 + H_3)\frac{\dot{\phi}}{\phi} = \frac{8\pi\rho}{\phi}, \quad (3.4)$$

$$B_x\left[A_y - B_y\right] + A_xB_y - B_{xy} = 0, \quad (3.5)$$

$$B_xA_z - A_{xz} = 0, \quad (3.6)$$

$$B_yA_z - A_{yz} = 0, = \quad (3.7)$$

$$B_x[H_1 - H_3] - (H_1)_x - (H_3)_x = 0, \quad (3.8)$$

$$B_y[H_1 - H_3] - (H_1)_y - (H_3)_y = 0, \quad (3.9)$$

$$A_z[H_1 - H_3] - (H_1)_z = 0. \quad (3.10)$$

From Equations (3.6), (3.7), and (3.10) after differentiating with respect to x, y, and t respectively we say,

$$(e^{-B}A_z)_x = (e^{-B}A_z)_y = 0, \text{and } (e^{A-B}A_z)_t = 0 \quad (3.11)$$

Using the integrability condition, the first two Equations of (3.11) imply that,

$$e^{-B}A_z = u(z,t)\,. \quad (3.12)$$

The cases $u = 0$ and $u \neq 0$ have to be considered separately because the integration proceeds in a different way in each case, and the limit $A_z \to 0$ of the solution for $A_z \neq 0$ is singular. Thus, if $A_z \neq 0$ we must have $(H_1)_{xy} = 0$, we shall consider the following possibilities

(1) $A_z = 0$,
(2) $A_z \neq 0$, $(H_1)_{xy} = 0$,

to get solutions of the field Equations.

3. 1 THE SUBFAMILY $A_z = 0$

The Equations (3.6), (3.7), and (3.10) are equal zero and fulfilled identically. In solving the other Equations, we can assume that $(H_1)_x = (H_1)_y = 0$ because otherwise, the Equations have no solutions; [11]. Then

$$e^A = \Phi(t)e^{\nu(x,y)}\,, \quad (3.13)$$

where Φ and ν are unknown functions, while Equations (3.8), and (3.9) are equal zero imply that

$$e^{B-A}B_x = \tilde{\eta}_1(x,y,z), e^{B-A}B_y = \tilde{\eta}_2(x,y,z), \quad (3.14)$$

where $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are other unknown functions. Using Equation (3.13) for e^A, and denoting

$$\tilde{\eta}_1(x,y,z) = \eta_1 e^{-\nu}, \tilde{\eta}_2(x,y,z) = \eta_2 e^{-\nu}\,, \quad (3.15)$$

we obtain

$$e^B B_x = \Phi(t)\eta_1(x,y,z), \quad e^B B_y\Phi(t)\eta_2(x,y,z). \quad (3.16)$$

The integrability condition $(e^B B_x)_y = \left(e^B B_y\right)_x$ implies $(\eta_1)_y = (\eta_2)_x$. This means that a function $\eta(x,y,z)$ exists such that $\eta_1=\eta_x, \eta_2=\eta_y$.Knowing this, (3.14) can be integrated to give

$$e^B = \Phi(t)\eta(x,y,z)+\mu(z,t). \quad (3.17)$$

where μ is an unknown function. Now if we replace A , and B from Equations (3.13), and (3.17) in the field Equation (3.3), we have similar differential Equations in Φ as

$$2\Phi\ddot{\Phi}+\dot{\Phi}^2+2\Phi\dot{\Phi}\left(\frac{\dot{\phi}}{\phi}\right) +\Phi^2\left[\frac{8\pi p}{\phi}+\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{\ddot{\phi}}{\phi}\right]=-K, \quad (3.18)$$

where Φ, and p depend only on t, and K an arbitrary constant [6],

$$K=-e^{-2\nu}\left[\nu_{xx}+\nu_{yy}\right], \quad (3.19)$$

because ν depends only on x, and y. Here we take the solution for ν in the form

$$e^{-\nu}=\alpha(z)(x^2+y^2)+\beta_1(z)x+\beta_2(z)y+\gamma(z), \quad (3.20)$$

with the restriction

$$\beta_1^2+\beta_2^2-4\alpha\gamma=-K, \quad (3.21)$$

where $\alpha(z), \beta_1(z), \beta_2(z)$, and $\gamma(z)$ are arbitrary functions (α, and γ being real) Now to determine the function η, we have from the field Equations (3.1), and (3.2), the solution

$$(e^{-\nu}\eta)_{xx}=(e^{-\nu}\eta)_{yy}=0. \quad (3.22)$$

From the field Equation (3.5), we have the solution

$$e^{-\nu}\eta=P(z)[x^2+y^2]+Q_1(z)x+Q_2(z)y+S(z), \quad (3.23)$$

where $P(z), Q_1(z), Q_2(z)$, and $S(z)$ are arbitrary functions. Also the metric (2.1) can be written as

$$ds^2=dt^2-\Phi(t)^2e^{2\nu(x,y)}(dx^2+dy^2)-(\Phi(t)\eta(x,y,z)+\mu(z,t))^2dz^2. \quad (3.24)$$

The average scale-factor $a(t)$, spatial volume V, Mean HP, and deceleration parameter are

$$V=a^3=\sqrt{-g}=e^{2\nu}\eta\mu\Phi^3, H=\frac{\dot{\Phi}}{\Phi}+\frac{\dot{\mu}}{3\mu}, \quad (3.25)$$

$$q=\frac{-\mu\left[6\mu\dot{\Phi}^2+3\mu\Phi\ddot{\Phi}+6\Phi\dot{\Phi}\dot{\mu}+\Phi^2\ddot{\mu}\right]}{\left(3\mu\dot{\Phi}+\Phi\dot{\mu}\right)^2}, \quad (3.26)$$

in which HPs in the directions of x, y, and z axes are,

$$H_1=H_2=\frac{\dot{\Phi}}{\Phi}, H_3=\frac{\dot{\Phi}}{\Phi}+\frac{\dot{\mu}}{\mu}. \quad (3.27)$$

The cosmological parameters such as the scalar expansion θ, and shear scalar σ are given by,

$$\theta=\frac{3\dot{\Phi}}{\Phi}+\frac{\dot{\mu}}{\mu}, \sigma=\frac{1}{\sqrt{3}}\frac{\dot{\mu}}{\mu}. \quad (3.28)$$

Using Equations (3.1) to (3.4), we have the expression for density as

$$\rho=\frac{-\phi}{8\pi}\left[6\frac{\ddot{\Phi}}{\Phi}-\left(\frac{\dot{\mu}}{\mu}\right)^2+2\frac{\ddot{\mu}}{\mu}+\frac{4\dot{\Phi}\dot{\mu}}{\Phi\mu}-3p+2\omega\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{3\ddot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{3\dot{\Phi}}{\Phi}+\frac{\dot{\mu}}{\mu}\right)\right]. \quad (3.29)$$

An addition singularity of infinite density occurs where (and if) $e^B=0$. Krolak et al [5], showed that the Big-Bang (BB) singularity at $\Phi=0$ in the $K=0$ subcase of IS solution is a naked strong curvature singularity. Therefore, the SSs are another counter example to the oldest and simplest formulation of the cosmic censorship hypothesis. In fact, Krolak et al [5] result shows that the SSs are not sufficiently generic from the point of view of the cosmic censorship paradigm. The shell-crossing singularity (the one at, $\Phi_z=0$), although naked as well, is not strong. Several other papers have been published in which the spherically symmetric limit of the $A_z\neq 0$ SSs (i.e. the Lemaitre-Tolman model) has been discussed as a testing ground for cosmic censorship [9].

The computations of relevant tensors; for metric (3.24) gives,

$$R = 4\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + 4\left(\frac{\ddot{\Phi}}{\Phi}\right) + 8\frac{\dot{\Phi}\dot{\mu}}{\Phi\mu} + 2\frac{\ddot{\mu}}{\mu} + 2\frac{e^{-2\nu}}{\eta\Phi^2}\left[\eta(\nu_{xx} + \nu_{yy}) + \eta_{xx} + \eta_{yy}\right]. \quad (3.30)$$

We consider the hyperbolic counterparts of these space-times even though they are of Bianchi type-III, ifwe take $\beta_1 = \beta_2 = 0, \alpha = \frac{K}{4}$, and $\gamma = 1$, the Equation (3.20) becomes

$$e^{-\nu} = 1 + \frac{K}{4}(x^2 + y^2). \quad (3.31)$$

Further simplifications may be introduced by coordinate transformations one of these if $\alpha = \beta_2 = \gamma = 0$, and $\beta_1 = 1$, then the Equation (3.23) becomes $\eta = e^{\nu}x$, so the metric (3.24) becomes,

$$ds^2 = dt^2 - \left[\frac{\Phi(t)}{1 + \frac{K}{4}(x^2 + y^2)}\right]^2 (dx^2 + dy^2 + x^2 dz^2). \quad (3.32)$$

Using the spatial transformation

$$x = r sin\theta, y = r cos\theta, z = \phi.$$

The metric (3.32) reduces to,

$$ds^2 = dt^2 - \left[\frac{\Phi(t)}{1 + \frac{K}{4}r^2}\right]^2 \left(dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right), \quad (3.33)$$

where $r^2 = x^2 + y^2$, and the metric is group of G_3. This form metric is similar to FLRW metric.

(1) when $K = 0$, (flat) the metric is Bianchi type-I, and Bianchi type- VII_0,
(2) when $K = +1$, (closed) the metric is Bianchi type-IX,
(3) when $K = -1$, (open) the metric is Bianchi type-V, and Bianchi type-VII_h,

Introducing complex variables for convenience

$$\xi = x + iy, \bar{\xi} = x - iy,$$

in which the Equation (3.19) becomes

$$4e^{-2\nu}\nu_{\xi\bar{\xi}} = -K. \quad (3.34)$$

Differentiating this by ξ we obtain,

$$\left(\nu_{\xi\xi} - \nu_\xi^2\right)_{\bar{\xi}} = 0,$$

and hence without loss of generality, we take

$$\nu_{\xi\xi} - \nu_\xi^2 = 0. \quad (3.35)$$

This implies

$$(e^{-\nu})_{\xi\xi} = (e^{-\nu})_{\bar{\xi}\bar{\xi}} = 0, \quad (3.36)$$

since ν, is real,

$$e^{-\nu} = \alpha\xi\bar{\xi} + \beta_1\xi + \bar{\beta}_2\bar{\xi} + \gamma, \quad (3.37)$$

so by substitution in Equation (3.34) implies

$$\alpha\gamma - \beta_1\bar{\beta}_2 = \frac{K}{4}, \quad (3.38)$$

if we take $\beta_1 = \beta_2 = 0, \alpha = \frac{K}{2}$, and $\gamma = \frac{1}{2}$, the Equation (3.37) becomes

$$e^{-\nu} = \frac{1}{2}\left(1 + K\xi\bar{\xi}\right). \quad (3.39)$$

Now to determine the function η, from the field Equation (3.5) we have,

$$(e^{-\nu}\eta)_{\xi\xi} = (e^{-\nu}\eta)_{\bar{\xi}\xi} = 0. \quad (3.40)$$

Now from the field Equations (3.1), and (3.2), we have the solution

$$e^{-\nu}\eta = \alpha(z)\xi\bar{\xi} + \beta_1(z)\frac{\xi + \bar{\xi}}{2} + \beta_2(z)\frac{\xi - \bar{\xi}}{2i} + \gamma(z), \quad (3.41)$$

with the restriction as gives in Equation (3.21). Further simplifications may be introduced by coordinate transformations one of these if $\alpha = \beta_2 = \gamma = 0$, and $\beta_1 = 1$, then the Equation (3.41) becomes, $\eta = \frac{\xi + \bar{\xi}}{2}e^{\nu}$. Also the metric (2.1) can be written as

$$ds^2 = dt^2 - \left[\frac{\Phi(t)}{1+\frac{K}{4}\xi\bar{\xi}}\right]^2 \left(d\xi d\bar{\xi} + \left(\frac{\xi+\bar{\xi}}{2}\right)^2 dz^2\right). \quad (3.42)$$

This metric is a general axially symmetric space-time.

3. 2 THE SUBFAMILY $A_z \neq 0, (H_1)_{xy} = 0$

This case gives useful information in astrophysics, and cosmology. From Equation (3.12), it is clear that $u(z,t) \neq 0$, hence we write,

$$e^B = \frac{A_z}{u(z,t)}. \quad (3.43)$$

Using Equation (3.43) in Equation (3.11), we get

$$(e^{A-B}A_z)_t = (e^A u)_t = 0. \quad (3.44)$$

The solution of which can be given as,

$$e^A = \Phi(z,t)e^{\nu(x,y,z)}, \quad (3.45)$$

where $\Phi(z,t) = \frac{1}{u}$, it follows that $(H_1)_x = (H_1)_y = 0$, so Equation (3.43) implies that

$$e^B = \Phi(z,t)A_z. \quad (3.46)$$

The arbitrary factor dependent on z can be introduced in e^B by a transformation of the form $z = f(z')$. This will simplify the limiting transition to the RW models. Thus,

$$e^B = h(z)\Phi(z,t)A_z = h(z)(\Phi_z + \Phi\nu_z). \quad (3.47)$$

The evolution Equation for Φ gives

$$2\Phi\ddot{\Phi} + \dot{\Phi}^2 + 2\Phi\dot{\Phi}\left(\frac{\dot{\phi}}{\phi}\right) + \Phi^2\left[\frac{8\pi p}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi}\right] = -K(z), \quad (3.48)$$

The function ν satisfies

$$e^{-2\nu}[\nu_{xx} + \nu_{yy}] - D^2 = K(z), \quad (3.49)$$

where $D = \frac{1}{h}$. Following the procedure adopted by Szafron [14] discussed in [9]. we may write the solutions in the form

$$e^{-\nu} = \alpha(z)[x^2 + y^2] + \beta_1(z)x + \beta_2(z)y\,\gamma(z), \quad (3.50)$$

with the restriction

$$\beta_1^2 + \beta_2^2 - \alpha\gamma = \frac{-1}{4}(D^2 + K(z)). \quad (3.51)$$

The first integral of the Equation (3.48), may be given by

$$\dot{\Phi}^2 = \frac{M}{\Phi} + K(z) - \frac{1}{\Phi}\int \dot{\Phi}\frac{\partial}{\partial t}(\Phi^2)\left(\frac{\dot{\phi}}{\phi}\right)dt - \frac{1}{3\Phi}\int \frac{\partial}{\partial t}(\Phi^3)\left[\frac{8\pi p}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi}\right]dt. \quad (3.52)$$

Now using conservation Equation (1.3) we get

$$\dot{\rho} + (H_3 + 2H_1)(\rho + 2p) = 0. \quad (3.53)$$

If we further assume that the perfect fluid obeys the barotropic Equation of state of the form

$$p = \omega\rho, \qquad 0 \leq \omega \leq 1,$$

with the equation of state parameter ω as time-independent. In this case, Equation (3.53) can be integrated for the energy density to yield

$$\rho = \left(\frac{1}{a}\right)^{3(1+2\omega)}. \quad (3.54)$$

In this model, the BB is not simultaneous in the comoving, and synchronous time t. With this t, the BB is a process extended in time rather than a single event in space-time.

The ISCM that could model our universe began with a BB a moment in time at which the scale factor $a(t)$ vanishes, and the geometry of the universe is singular. The singular nature of the BB is apparent from Equation (3.54). The densities of matter, and radiation are infinite when $a = 0$ [9].

The BB occurred at every place in space at one moment in time. The notion of a geometry of spaceï¿½time breaks down at a singularity, along with the predictive power of the law of geometry, such as Einstein’s Equation As far as making predictions in physic, concerned, the universe began at the BB. For this reason, the BB is conventionally assigned the time $t = 0$. Also the metric (2.1) can be written as

$$ds^2 = dt^2 - \Phi^2 e^{2\nu}(dx^2 + dy^2) - (h\Phi A_z)^2 dz^2. \quad (3.55)$$

The average scale-factor $a(t)$, spatial volume V, and Mean HP are

$$V = a^3 = \sqrt{-g} = e^{2\nu}\, h^2\, \Phi^2\, (\Phi_z + \Phi\, \nu_z), \quad (3.56)$$

$$H = \frac{\Phi\,\dot{\Phi}_z + \dot{\Phi}\,(2\,\Phi_z + 3\,\Phi\,\nu_z)}{3\,\Phi\,(\Phi_z + 3\,\Phi\,\nu_z)}, \quad (3.57)$$

in which HPs in the directions of x, y, and z axes are,

$$H_1 = H_2 = \frac{\dot{\Phi}}{\Phi},\ H_3 = \frac{\dot{\Phi}_z + \dot{\Phi}\,\nu_z}{\Phi_z + \Phi\,\nu_z}. \quad (3.58)$$

The cosmological parameters such as the scalar expansion (θ), and shear scalar (σ) are given by,

$$\theta = \frac{\Phi\,\dot{\Phi}_z + \dot{\Phi}\,(2\,\Phi_z + 3\,\Phi\,\nu_z)}{\Phi\,(\Phi_z + 3\,\Phi\,\nu_z)}, \quad (3.59)$$

$$\sigma = \frac{\dot{\Phi}\,\Phi_z - \Phi\,\dot{\Phi}_z}{\sqrt{3}\,\Phi\left(\dot{\Phi} + \Phi\,\nu_z\right)}, \quad (3.60)$$

From Equations (3.59) and (3.60), we obtain

$$\frac{\sigma^2}{\theta^2} = \frac{\left(\dot{\Phi}\,\Phi_z - \Phi\,\dot{\Phi}_z\right)^2}{3\,\left(\Phi\,\dot{\Phi}_z + \dot{\Phi}\,(2\,\Phi_z + 3\,\Phi\,\nu_z)\right)} = constant. \quad (3.61)$$

Since, $\frac{\sigma}{\theta}$ is a non-zero constant the model does not approach to isotropy.

For the model (3.55), and by using Equation (3.54), the energy density ρ becomes

$$\rho = \left(\frac{e^{-2\nu}}{h^2\,\Phi^2\,(\Phi_z + 3\,\Phi\,\nu_z)}\right)^{1+2\omega},\ 0\ \omega \le 1 \quad (3.62)$$

The sign of one of them fixes the geometry of the $t = constant$ 3-surfaces, and the type of evolutions are elliptic, parabolic, and hyperbolic. The sign of another function determines the geometry of the $t = constant$, and $z = constant$, 2-surfaces they quasi-spherical, quasi-plane, and quasi-hyperbolic models. Only the quasi-spherical model has been found useful in astrophysical cosmology, thus our results match with [10].

In the exact IS can be employed not only for studying the dynamics, and the geometry of the universe, but also to investigate the formation, and evolution of structures.

4. CONCLUSION

In this paper, we have considered that the ISC solutions with perfect fluid as the matter distribution. We can classify the solutions into two categories namely, (1) $A_z = 0$, and (2) $A_z \neq 0$. The first set of solutions are known as a quasi-spherical solution while the second class of solution is termed as a cylindrical type of solutions.

Two properties have already been mentioned, the lack of any symmetry in general, and the existence of the surfaces of constant curvature $t = z = constant$. In fact, the lack of symmetry was proved by Bonnor, Sulaiman, and Tomimura(1977) [3] for the Szekeres solutions, but since they assume $p = 0$ of the Szafron space-times, it follows immediately that the latter has, in general, no symmetry either. Other properties in common are as follows, thus properties were discussed in [9].

The Weyl tensor of the Szafron space-times has its magnetic part with respect to the velocity field of the source equal to zero (Szafron, and Collins 1979 [7], Barnes and Rowlingson 1989 [1]), and is in general of Petrov type D (Szafron 1977). It degenerates to zero in the FLRW limit only.

The slices $t = constant$ of these space-times are conformally flat (Berger, Eardley, and Olson 1977 [2]). This indicates that the space-times are non-radiative in the sense of York (1972) [16].

Note that the curvature of the xy-surfaces is a global constant only in the $A_z = 0$ subfamily, and there it is equal to the curvature index of the $t =$

$constant$ slices in the resulting FLRW limit. In the $A_z \neq 0$ subfamily, the curvature of the xy-surfaces is determined by $\Delta = \beta_1^2 + \beta_2^2 - \alpha\gamma$ and is independent of the curvature index of the FLRW limit, which is determined by $K(z)$. Both $K(z)$, and Δ are only constant within each xy-surface and can vary within $t = constant$ slice. The variation of K over $t = constant$ slice has an interesting consequence: Ifobservers in different spatial locations of the Szafron $A_z \neq 0$ Universe are trying to approximate it by FLRW models, then each of them may choose a different FLRW model. Even the sign of K is not a global property of a general Universe. Its global constancy is a peculiarity of the FLRW class.

The Szafron space-times trivialize: those with $A_z \neq 0$ become FLRW, and those with $A_z = 0$ acquire either FLRW or K-S geometry or the plane and hyperbolic counterparts of the latter (Spero and Szafron 1978 [13]).

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