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MATHEMATICAL INVESTIGATION OF COUNTER-CURRENT IMBIBITION PHENOMENON IN HETEROGENEOUS POROUS MEDIUM

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ABSTRACT

The counter-current imbibition phenomenon in heterogeneous porous medium which arises during secondary oil recovery process in petroleum reservoirs is discussed. An optimal homotopy analysis method is applied to obtain the approximate analytical solution of the governing nonlinear partial differential equation with appropriate boundary conditions. Also the counter-current imbibition phenomenon is derived if the porous medium is homogeneous. The convergence of the approximate analytical solution is decided by using averaged squared residual of governing equation. The numerical values are obtained for the saturation of injected water and saturation profiles are plotted using Mathematica in heterogeneous as well as homogeneous porous medium.

Keywords: Counter-current imbibition, Darcy law, Conservation of mass, Squared residual.

AMS subject classification: 76T99, 76S05, 76A02, 78M50.

INTRODUCTION

Fractured reservoirs are important petroleum resources. They are composed of two continua: the fracture network and matrix. The fractures are highly permeable as compared to the matrix whose permeability may be several orders of lower magnitude. But majority of the recoverable oil is contained in the matrix network. Fracure network has very less amount of oil in it. Waterflooding is frequently implemented to increase recovery in fractured reservoirs.

The performance of waterflooding depends crucially on the wettability of the reservoir. If the reservoir is oil-wet, water will not readily displace oil in the matrix and only the oil in the fractures will be displaced, resulting in poor recoveries. In water-wet fractured reservoirs, imbibition can lead to significant recoveries. Imbibition is the mechanism of displacement of non-wetting phase by wetting phase. Strong capillary forces lead to the imbibition of water as the wetting phase into the matrix and the discharged oil is displaced into the fractures.

Imbibition can take place by co-current and/or counter-current flow. In co-current flow the water and oil flow in the same direction, and water pushes oil out of the matrix. In counter-current flow, the oil and water flow in opposite directions, and oil escapes by flowing back along the same direction along which water has imbibed. Co-current imbibition is faster and can be more efficient than counter-current imbibition but counter-current imbibition is often the only possible displacement mechanism for cases where a region of the matrix is completely surrounded by water in the fractures.

Counter-current imbibition phenomenon is discussed by many researchers. Cocurrent and countercurrent imbibition in a water-wet matrix block is discussed by Pooladi-Darvish and Firoozabadi [1]. Through a detailed study of the governing equations and boundary conditions they have provided significant insight into the mathematical and physical differences between co- and counter-current imbibition. Experimental study of co-current and counter-current flows in natural porous media is done by Bourbiaux and Kalaydjian [2]. Behbahani et al. [3] have performed the fine grid, one- and two-dimensional simulations of counter-current imbibition and have compared the results with experimental measurements in the literature. Reis and Cil [4] have developed a model for oil expulsion by counter-current water imbibition in rocks. Patel et al. [5] has obtained a homotopy series solution to a nonlinear partial differential equation arising from a mathematical model of the counter–current imbibition phenomenon in a heterogeneous porous medium. Patel and Meher [6] have discussed the counter-current imbibition phenomenon in heterogeneous porous media with gravitational and inclination effect and have calculated the saturation rate of wetting phase by using Adomian decomposition method. Patel and Desai [7] have obtained approximate analytical solution for countercurrent imbibition phenomenon in inclined homogeneous porous medium.

In this work we have discussed one dimensional counter-current imbibition in heterogeneous porous medium and the approximate analytical expression for the solution of the governing nonlinear partial differential equation with appropriate boundary conditions is obtained by optimal homotopy analysis method. Also we derive the approximate analytical solution in the case if the porous medium is homogeneous. Fluids (water and oil) are assumed immiscible and incompressible, and the porous solid matrix is assumed to be rigid.

AIM AND SCOPE

The aim of this work is to obtain the convergent approximate analytical solution for the saturation of injected water as a function of position and time by optimal homotopy analysis method. This type of solution can help to predict the amount of water required to inject for recovering oil from the petroleum reservoir during secondary oil recovery process. Hence this type of mathematical investigation is useful for predicting oil recovery from petroleum reservoir.

PROBLEM FORMULATION

We have considered a fully oil saturated sample of rectangular porous matrix of length l with only one permeable side which is exposed to an adjacent formation of the injected fluid (i) as water as shown in Fig.1. The open side of the matrix is labeled as imbibition face.

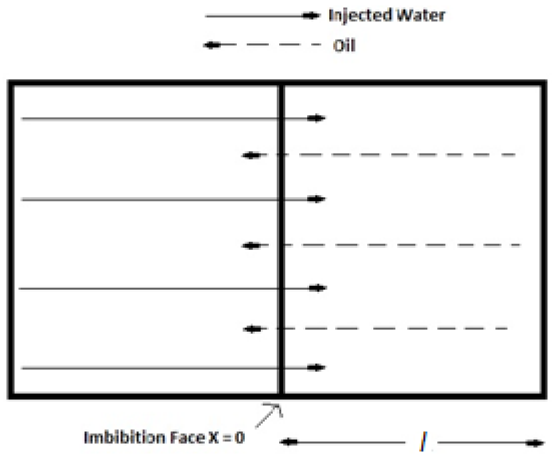

Fig.1. Schematic of 1D counter-current imbibition

This arrangement gives rise to the phenomenon of linear counter-current imbibition that is a spontaneous flow of water into the porous matrix and a counter flow of oil from the matrix.

Assuming the validity of Darcy's law, the volumetric flow rates (Darcy velocities) V_i and V_n of water and oil can be written as [19]

$$V_i = -K(x)\frac{k_i}{\mu_i}\frac{\partial p_i}{\partial x} \quad (1)$$

$$V_n = -K(x)\frac{k_n}{\mu_n}\frac{\partial p_n}{\partial x} \quad (2)$$

where $K(x)$ is the permeability of the heterogeneous medium which is nonconstant function of position, k_i and k_n are the relative permeabilities of water and oil respectively, μ_i and μ_n are the constant viscosities of water and oil respectively, p_i and p_n are the pressures of water and oil respectively.

Assuming the absence of any source and/or sink term, conservation mass equation for water volume can be written as [19,21]

$$P(x)\frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \quad (3)$$

where $P(x)$ is the porosity of the heterogeneous porous matrix which is nonconstant function of position.

In counter-current imbibition phenomenon, oil and water are flowing in opposite direction. We assume that the sum of their velocities is zero.[26] i.e.

$$V_i + V_n = 0 \tag{4}$$

The capillary pressure p_c generally is expressed as the pressure in the nonwetting phase minus the pressure in the wetting phase, and so commonly a positive value. We define the water-oil capillary pressure as [21]

$$p_c(S_i) = p_n - p_i \tag{5}$$

Since the capillary pressure is a function of the phase saturation, we consider [27]

$$p_c = -\beta S_i \tag{6}$$

where β is a positive capillary pressure parameter.

For definiteness of the mathematical analysis, the relationship between phase saturation and relative permeability as given by Scheidegger and Johnson (1961) is used here [11].

$$k_i = S_i \tag{7}$$

According to Oroveanu [10], the porosity and permeability of the heterogeneous porous medium are:

$$P(x) = \frac{1}{a - bx} \tag{8}$$

and

$$K(x) = K_c P(x) \tag{9}$$

where $a - bx \geq 1 \quad for \quad all \quad x \in [0, l]$ and a, b and K_c are constants.

Using (1) and (2) in (4), we have

$$\frac{k_i}{\mu_i} K(x) \frac{\partial p_i}{\partial x} + \frac{k_n}{\mu_n} K(x) \frac{\partial p_n}{\partial x} = 0 \tag{10}$$

Using (5), this becomes

$$\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n} \right) \frac{\partial p_i}{\partial x} + \frac{k_n}{\mu_n} \frac{\partial p_c}{\partial x} = 0 \tag{11}$$

Simplyfying, we obtain

$$\frac{\partial p_i}{\partial x} = -\frac{k_n}{\mu_n} \left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n} \right)^{-1} \frac{\partial p_c}{\partial x} \tag{12}$$

On substituting the value of $\frac{\partial p_i}{\partial x}$ in (1), we get

$$V_i = K(x) \frac{k_i}{\mu_i} \frac{k_n}{\mu_n} \left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n} \right)^{-1} \frac{\partial p_c}{\partial x} \tag{13}$$

Using (13) in (3), we get

$$P \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x} \left[K(x) \frac{k_i}{\mu_i} \frac{k_n}{\mu_n} \left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n} \right)^{-1} \frac{\partial p_c}{\partial x} \right] = 0 \tag{14}$$

According to Scheidegger [21], it is assumed that

$$\frac{k_i}{\mu_i} \frac{k_n}{\mu_n} \left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n} \right)^{-1} \approx \frac{k_i}{\mu_i}$$

Hence (14) reduces to

$$P \frac{\partial S_i}{\partial t} + \frac{\partial}{\partial x} \left[K(x) \frac{k_i}{\mu_i} \frac{\partial p_c}{\partial S_i} \frac{\partial S_i}{\partial x} \right] = 0 \tag{15}$$

Using (6), (7) and (9) into (15), we get

$$P(x) \frac{\partial S_i}{\partial t} - \frac{\beta K_c}{\mu_i} \frac{\partial}{\partial x} \left[S_i P(x) \frac{\partial S_i}{\partial x} \right] = 0$$

which reduces to

$$\frac{\partial S_i}{\partial t} - \frac{\beta K_c}{\mu_i} \left[\begin{array}{l} S_i \frac{\partial^2 S_i}{\partial x^2} + S_i \frac{\partial S_i}{\partial x} \frac{1}{P} \frac{\partial P}{\partial x} \\ + \left(\frac{\partial S_i}{\partial x} \right)^2 \end{array} \right] = 0 \tag{16}$$

Using dimensionless variables

$$X = \frac{x}{l}, \quad T = \frac{\beta K_c}{\mu_i l^2} t$$

(16) reduces to

$$\frac{\partial S_i}{\partial T} - S_i \frac{\partial^2 S_i}{\partial X^2} - S_i \frac{\partial S_i}{\partial X} \frac{1}{P} \frac{\partial P}{\partial X} - \left(\frac{\partial S_i}{\partial X} \right)^2 = 0 \tag{17}$$

Using binomial theorem, we can simplify $\frac{1}{P} \frac{\partial P}{\partial X}$ as follows

$$\frac{1}{P}\frac{\partial P}{\partial X} = \frac{\partial}{\partial X}\left[\ln P\right] = \frac{\partial}{\partial X}\left[\frac{blX}{a} - \ln a\right] = \frac{bl}{a}$$

Using this (17) reduces to

$$\frac{\partial S_i}{\partial T} - S_i\frac{\partial^2 S_i}{\partial X^2} - \frac{bl}{a}S_i\frac{\partial S_i}{\partial X} - \left(\frac{\partial S_i}{\partial X}\right)^2 = 0 \tag{18}$$

Eq. (18) is nonlinear partial differential equation governing the counter-current imbibition in heterogeneous porous medium. The solution of this equation represents the saturation of injected water.

Let at the imbibition face, the saturation of injected water be linear function of time, that is

$$S_i(0,T) = \alpha T \quad for \quad T > 0 \tag{19}$$

where α is constant.

Since, it is assumed that the porous matrix is completely surrounded by an impermeable surface except for one end, we consider

$$\frac{\partial S_i}{\partial X}(1,T) = 0 \quad for \quad T > 0 \tag{20}$$

We solve equation (18) together with boundary conditions (19) and (20) using optimal homotopy analysis method [12-18,22-25].

SOLUTION OF THE PROBLEM USING OPTIMAL HOMOTOPY ANALYSIS METHOD

We choose

$$S_{i_0}(X,T) = \alpha T\left[e^{-X} + Xe^{-1}\right] \tag{21}$$

as the initial approximation of $S_i(X,T)$ which satisfies boundary conditions (19) and (20). We select the linear operator as

$$\mathrm{L}[\phi(X,T;q)] = \frac{\partial^2\phi(X,T;q)}{\partial X^2} \tag{22}$$

with the property

$$\mathrm{L}[f] = 0 \quad when \quad f = 0. \tag{23}$$

where $q \in [0,1]$ is the embedding parameter. Furthermore, based on governing equation (18), a nonlinear operator is defined as

$$\begin{aligned}\mathrm{N}[\phi(X,T;q)] = {} & \frac{\partial\phi(X,T;q)}{\partial T} \\ & - \phi(X,T;q)\frac{\partial^2\phi(X,T;q)}{\partial X^2} \\ & - \frac{bl}{a}\phi(X,T;q)\frac{\partial\phi(X,T;q)}{\partial X} \\ & - \left\{\frac{\partial\phi(X,T;q)}{\partial X}\right\}^2\end{aligned} \tag{24}$$

Let c_0 denote a nonzero auxiliary parameter. The zeroth order deformation equation is [25]

$$\begin{aligned}&(1-q)\mathrm{L}[\phi(X,T;q) - S_{i_0}(X,T)] \\ &= c_0 qH(X,T)\mathrm{N}[\phi(X,T;q)]\end{aligned} \tag{25}$$

In (25), $H(X,T)$ is nonzero auxiliary function and $\phi(X,T;q)$ is an unknown function.

Obviously, when $q = 0$ and $q = 1$, we have from (23) and (25),

$$\phi(X,T;0) = S_{i_0}(X,T) \tag{26}$$

and $$\phi(X,T;1) = S_i(X,T) \tag{27}$$

Therefore, the solution $\phi(X,T;q)$ varies from the initial approximation $S_{i_0}(X,T)$ to the exact solution $S_i(X,T)$ of (18) as q increases from 0 to 1.

Obviously, $\phi(X,T;q)$ is determined by the auxiliary linear operator L, the initial guess $S_{i_0}(X,T)$ and the auxiliary parameter c_0 . They can be selected with great freedom..

Assuming that all of them are so properly selected such that the Maclaurin series

$$\phi(X,T;q) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T)q^m \tag{28}$$

exists and besides converges at $q = 1$, we have the homotopy-series solution

$$S_i(X,T) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T) \tag{29}$$

where $S_{i_m}(X,T) = \frac{1}{m!}\frac{\partial^m \phi(X,T;q)}{\partial q^m}\Big|_{q=0}$ (30)

Differentiating the zeroth order deformation equations (25) m times with respect to the embedding parameter q and then dividing them by $m!$ and finally setting $q=0$ i.e. in case of no embedding, we have the so called high order deformation equations

$$\mathrm{L}[S_{i_m}(X,T) - \chi_m S_{i_{m-1}}(X,T)] = c_0 H(X,T) R_m(X,T) \quad (31)$$

subject to the boundary conditions

$$S_{i_m}(0,T) = 0, \quad \frac{\partial S_{i_m}}{\partial X}(1,T) = 0, \quad m \geq 1 \quad (32)$$

where

$$R_m(X,T) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}[\phi(X,T;q)]}{\partial q^{m-1}}\Big|_{q=0}$$
$$= \frac{\partial S_{i_{m-1}}}{\partial T} - \frac{bl}{a}\sum_{k=0}^{m-1} S_{i_k}\frac{\partial S_{i_{m-1-k}}}{\partial X} - \sum_{k=0}^{m-1} S_{i_k}\frac{\partial^2 S_{i_{m-1-k}}}{\partial X^2} - \sum_{k=0}^{m-1}\frac{\partial S_{i_k}}{\partial X}\frac{\partial S_{i_{m-1-k}}}{\partial X}$$

and $\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases}$

For the sake of simplicity, we take $H(X,T) = 1$.

The equations (31) are second order nonhomogeneous ordinary linear differential equations with constant cofficients for all $m \geq 1$ and can be solved by symbolic computation software such as Mathematica. Thus we have converted the original nonlinear problem (18)-(19)-(20) into an infinite sequence of linear subproblems (31)-(32). Hence the approximate analytical solution to the governing nonlinear problem takes the form:

$$S_i(X,T) = \alpha T\left[e^{-X} + Xe^{-1}\right]$$
$$+ c_0[\alpha e^{-X} + \frac{\alpha e^{-1}}{6}X^3 - \frac{bl\alpha^2 e^{-2}}{6a}T^2 X^3$$
$$+ \left(\frac{bl}{a} - 2\right)\frac{\alpha^2}{4}T^2 e^{-2X} + \frac{bl\alpha^2 e^{-1}}{a}T^2 e^{-X}$$
$$+ \left(\frac{bl}{a} - 1\right)\alpha^2 e^{-1}T^2 Xe^{-X} + \frac{\alpha e^{-1}}{2}X$$
$$+ \frac{2bl\alpha^2 e^{-2}}{a}T^2 X - \frac{\alpha^2 e^{-2}}{2}T^2 X^2$$
$$- \left(\frac{bl}{a} - 2\right)\frac{\alpha^2}{4}T^2 - \alpha - 2\left(\frac{bl}{a} - 1\right)\alpha^2 e^{-1}T^2$$
$$+ \left(\frac{bl}{a} - 2\right)\alpha^2 e^{-1}T^2] + \Lambda$$

(33)

The optimal value of c_0 can be obtained by minimizing averaged squared residual E_m.

As given by Liao [23,24], the averaged squared residual at the m th order of approximation is

$$E_m = \frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathrm{N}\left[\sum_{n=0}^{m} S_{i_n}\left(\frac{i}{M},\frac{j}{N}\right)\right]\right\}^2$$

(34)

Since the squared residual E_m is a function of c_0, we can find the optimal value of c_0 from

$$\frac{dE_m(c_0)}{dc_0} = 0 \quad (35)$$

This optimization method for obtaining the optimal value of c_0 has been applied recently to a number of problems for nonlinear ordinary and partial differential equations by many researchers [12-18,22-24].

The optimal value of c_0 is found by the minimum of E_{15} using Mathematica. We find that E_{15} attains its minimum value

1.4341×10^{-6} at $c_0 = -0.09$ which we can notice in Fig.2 which shows the curve of averaged squared residual E_{15} versus c_0.

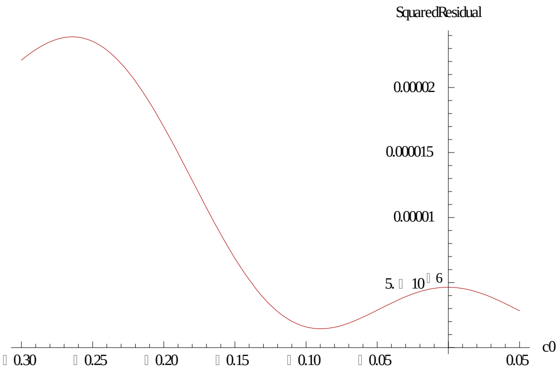

Fig.2. Averaged Squared Residual at fifteenth order of approximation versus convergence-control parameter

RESULTS AND DISCUSSION

The BVPh, a Mathematica package, is used to obtain numerical representation. Table 1 indicates the numerical values of saturation of injected water for different distance X and time T upto 15th order approximation using $c_0 = -0.09$. We have considered the unit length l of porous matrix, $a = 2$ and $b = 1$. Fig.3 represents the graph of $S_i(X,T)$ versus time T for fixed values of distance $X == 0.1, 0.2, \Lambda, 1$ .The lowermost graph corresponds to $X == 0.1$ and the uppermost corresponds to $X == 1$. Numerical values of Table 1 are used for Fig.3.

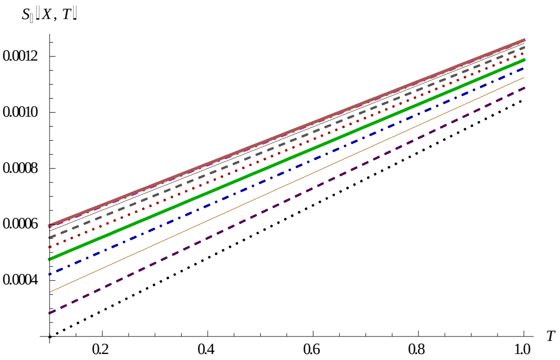

Fig 3. Saturation of injected water versus time for fixed values of distance

Derivation of Counter-current imbibition in homogeneous porous medium

If the porous medium is homogeneous, then its porosity is constant. Taking $b = 0$ in (8), the porosity $P =$ constant and (18) reduces to

$$\frac{\partial S_i}{\partial T} - S_i\frac{\partial^2 S_i}{\partial X^2} - \left(\frac{\partial S_i}{\partial X}\right)^2 = 0 \qquad (36)$$

which is the governing equation for counter-current imbibition phenomenon in homogeneous porous medium. The solution of (36) can be obtained from (33) by taking $b = 0$ which is as follows.

$$\begin{aligned} S_i(X,T) &= \alpha T\left[e^{-X} + Xe^{-1}\right] \\ &+ c_0[\alpha e^{-X} + \frac{\alpha e^{-1}}{6}X^3 - \frac{\alpha^2}{2}T^2e^{-2X} \\ &- \alpha^2 e^{-1}T^2Xe^{-X} + \frac{\alpha e^{-1}}{2}X - \frac{\alpha^2 e^{-2}}{2}T^2X^2 \\ &+ \frac{\alpha^2}{2}T^2 - \alpha] + \Lambda \end{aligned} \qquad (37)$$

This represents the approximate analytical expression for the saturation of injected water in homogeneous porous medium. The value of c_0 is obtained by minimizing averaged squared residual.

Fig.4 shows the curve of averaged squared residual E_{15} versus c_0 for homogeneous porous medium. The optimal value of c_0 is -0.21 corresponding to the minimum value 1.11576×10^{-6} of E_{15}.

The numerical values of saturation of water in the case of homogeneous porous medium are obtained upto 15th order approximation using using $c_0 = -0.21$ which are as shown in Table 2. Fig.5 shows the saturation profile of injected water versus time for fixed values of distance $X == 0.1, 0.2, \Lambda, 1$ upto 15th order approximation. The lowermost graph corresponds to $X == 0.1$ and the uppermost corresponds to $X == 1$.

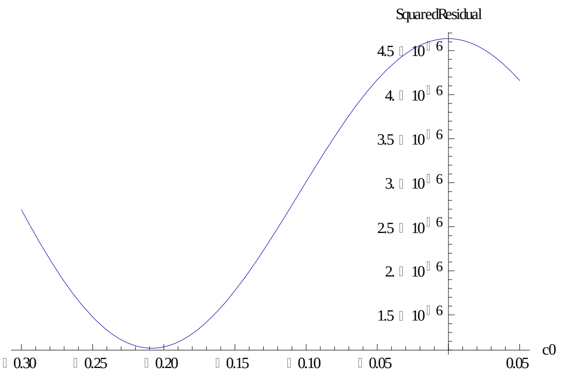

Fig.4. Averaged Squared Residual at fifteenth order of approximation versus convergence-control parameter for homogeneous porous medium

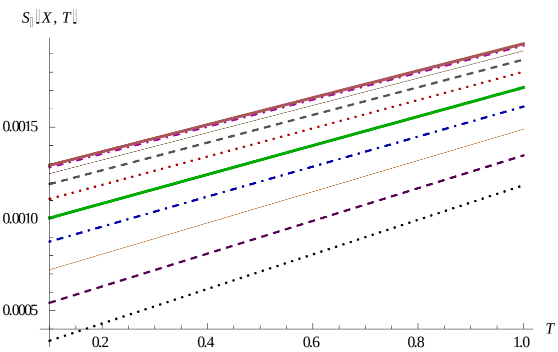

Fig.5. Saturation of injected water versus time for fixed values of distance in homogeneous porous medium

CONCLUSION

One-dimensional counter-current imbibition has been investigated, in which a non-wetting phase (oil) is displaced by a wetting phase (water). The optimal homotopy analysis method has been used to develop an approximate analytical expressions for the saturation of wetting phase (water) in heterogeneous and homogeneous porous medium.

At fixed location, the saturation of water increases as time increases which indicates that the oil will be displaced from the reservoir.

It has been found that the saturation of water increases fast in homogeneous porous medium as compared to the heterogeneous porous medium. So the oil recovery from homogeneous porous medium is fast than the oil recovery from heterogeneous medium.

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TABLES

Table 1: Numerical values of saturation of injected water in heterogeneous porous medium

X	T=0.1	T =0.2	T =0.3	T =0.4	T =0.5	T=0.6	T =0.7	T =0.8	T =0.9	T=1
0.1	1.97715E-4	2.91878E-4	3.8604E-4	4.80202E-4	5.74363E-4	6.68524E-4	7.62685E-4	8.56845E-4	9.51004E-4	1.04516E-3
0.2	2.83615E-4	3.72846E-4	4.62076E-4	5.51305E-4	6.40533E-4	7.29761E-4	8.18987E-4	9.08213E-4	9.97438E-4	1.08666E-3
0.3	3.58281E-4	4.43399E-4	5.28516E-4	6.13632E-4	6.98746E-4	7.8386E-4	8.68972E-4	9.54082E-4	1.03919E-3	1.1243E-3
0.4	4.22189E-4	5.03936E-4	5.85682E-4	6.67426E-4	7.49168E-4	8.30909E-4	9.12648E-4	9.94385E-4	1.07612E-3	1.15786E-3
0.5	4.75726E-4	5.54773E-4	6.33817E-4	7.12861E-4	7.91902E-4	8.70941E-4	9.49978E-4	1.02901E-3	1.10805E-3	1.18708E-3
0.6	5.19192E-4	5.96145E-4	6.73097E-4	7.50046E-4	8.26993E-4	9.03938E-4	9.80881E-4	1.05782E-3	1.13476E-3	1.2117E-3
0.7	5.52812E-4	6.28221E-4	7.03629E-4	7.79034E-4	8.54436E-4	9.29837E-4	1.00523E-3	1.08063E-3	1.15602E-3	1.23141E-3
0.8	5.76743E-4	6.51105E-4	7.25466E-4	7.99824E-4	8.74179E-4	9.48532E-4	1.02288E-3	1.09723E-3	1.17157E-3	1.24592E-3
0.9	5.91078E-4	6.64844E-4	7.38607E-4	8.12368E-4	8.86126E-4	9.59881E-4	1.03363E-3	1.10738E-3	1.18113E-3	1.25488E-3
1	5.95857E-4	6.69433E-4	7.43006E-4	8.16576E-4	8.90143E-4	9.63708E-4	1.03727E-3	1.11083E-3	1.18439E-3	1.25794E-3

Table 2: Numerical values of saturation of injected water in homogeneous porous medium

X	T=0.1	T =0.2	T =0.3	T =0.4	T =0.5	T=0.6	T =0.7	T =0.8	T =0.9	T=1
0.1	3.35851E-3	4.30026E-4	5.24198E-4	6.18367E-4	7.12531E-4	8.06693E-4	9.0085E-4	9.95004E-4	1.08915E-3	1.1833E-3
0.2	5.42933E-4	6.32184E-4	7.21428E-4	8.10666E-4	8.99898E-4	9.89123E-4	1.07834E-3	1.16755E-3	1.25676E-3	1.34596E-3
0.3	7.2271E-4	8.07851E-4	8.92984E-4	9.78108E-4	1.06322E-3	1.14833E-3	1.23343E-3	1.31852E-4	1.40360E-3	1.48867E-3
0.4	8.76395E-4	9.58166E-4	1.03993E-3	1.12168E-3	1.20342E-3	1.28515E-3	1.36686E-3	1.44857E-3	1.53027E-3	1.16116
0.5	1.00498E-3	1.08405E-3	1.16311E-3	1.24215E-3	1.32119E-3	1.40021E-3	1.47922E-3	1.55822E-3	1.63721E-3	1.71618E-3
0.6	1.10924E-3	1.18622E-3	1.26318E-3	1.34013E-3	1.41707E-3	1.49399E-3	1.5709E-3	1.6478E-3	1.72469E-3	1.80156E-3
0.7	1.18979E-3	1.26522E-3	1.34064E-3	1.41604E-3	1.49143E-3	1.56681E-3	1.64217E-3	1.71752E-3	1.79285E-3	1.86817E-3
0.8	1.24706E-3	1.32144E-4	1.39581E-3	1.47016E-3	1.5445E-3	1.61883E-3	1.69314E-3	1.76744E-3	1.84172E-3	1.91599E-3
0.9	1.28132E-3	1.35511E-3	1.42888E-3	1.50263E-3	1.57637E-3	1.6501E-3	1.72381E-3	1.79751E-3	1.87119E-3	1.94486E-3
1	1.29273E-3	1.36633E-3	1.43991E-3	1.51347E-3	1.58702E-3	1.66056E-3	1.73408E-3	1.80758E-3	1.88108E-3	1.95455E-3