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Lanczos Potential for Weyl Metric

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ABSTRACT

The Weyl metric represents a static spacetime which is non-vacuum in nature and does not fall in any of the Petrov types. In this paper, we have obtained Lanczos potential for the Weyl metric.

Keywords: Lanczos Potential, Newman-Penrose Formalism, Weyl Metric.

INTRODUCTION

It is known that the Weyl curvature tensor C_{hijk} (gravitational field) can be generated by the covariant differentiation of a rank three tensor L_{ijk} through the equation ([10], [11], [17])

$$\begin{aligned} C_{hijk} &= L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} \\ &+ L_{(hk)}g_{ij} + L_{(ij)}g_{hk} - L_{(hj)}g_{ik} - L_{(ik)}g_{hj} \\ &+ \frac{2}{3}L^{pq}_{p;q}\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right), \end{aligned} \tag{1}$$

where

$$L_{ij} = L^{k}_{i\ j;k} - L^{k}_{i\ k;j}. \tag{2}$$

Equation (1) is known as Weyl-Lanczos relation and the tensor L_{ijk} is known as Lanczos potential. It satisfies the following properties

$$\begin{aligned} &L_{ijk} = -L_{jik,} \\ &L^{t}_{i\,t} = 0,\ (or\ g^{kl}L_{kil} = 0), \\ &L_{ijk} + L_{jki} + L_{kij} = 0, \\ &L_{ij}{}^{k} = 0. \end{aligned} \tag{3}$$

Due to the properties (3), the non-trivial components of the Lanczos potential tensor reduces to ten out of sixty four. For arbitrary spacetimes, it is difficult to solve Weyl-Lanczos relation (1), as they are non-linear in nature. However, Novello and Velloso have obtained Lanczos potential for perfect fluid spacetimes by considering different conditions on general observer quantities. Newman-Penrose (NP) form of Weyl-Lanczos relation has proved useful for finding Lanczos potential for vacuum spacetimes of particular Petrov types [6]. Due to involvement of tetrad components of Ricci tensors in the NP field equations, Lanczos potential for non-vacuum spacetimes becomes difficult. Thus, it will become helpful to find Lanczos potential for particular non-vacuum spacetimes and it will lead us to Lanczos potential for arbitrary spacetimes. Recently, Hasmani and Panchal [16] have obtained Lanczos potential for Vaidya metric and Van-Stockum metric. Many solutions of Weyl-Lanczos relations have been obtained [1]-[12],[18],[20],[21]. But, a large class of solutions is still uncovered especially for non-vacuum spacetimes. Thus, it is fruitful to find Lanczos potentials for unknown situations.

The Weyl metric is non-vacuum and none of the Petrov types. The Lanczos potential for this metric has been obtained and the Lanczos scalars have been expressed in term of spin coefficients. The necessary computations are carried out using Mathematica programs developed by Hasmani and co-workers [13]-[15].

WEYL METRIC

The Weyl metric [15], in cylindrical coordinates, is given by

$$ds^2 = \frac{1}{f}[e^{2\gamma}\left(dr^2 + dz^2\right) + r^2 d\varphi^2] - fdt^2, \tag{4}$$

where f and γ are real function of r and z.

The following is chosen null tetrad,

$$
\begin{aligned}
l^k &= \frac{1}{\sqrt{2}}\left(e^{-\gamma}\sqrt{f}\,\delta_1^k + \frac{1}{\sqrt{f}}\,\delta_4^k\right),\\
n^k &= \frac{1}{\sqrt{2}}\left(-e^{-\gamma}\sqrt{f}\,\delta_1^k + \frac{1}{\sqrt{f}}\,\delta_4^k\right),\\
m^k &= \frac{1}{\sqrt{2}}\left(e^{-\gamma}\sqrt{f}\,\delta_2^k + i\frac{\sqrt{f}}{r}\,\delta_3^k\right),\\
\overline{m}^k &= \frac{1}{\sqrt{2}}\left(e^{-\gamma}\sqrt{f}\,\delta_2^k - i\frac{\sqrt{f}}{r}\,\delta_3^k\right).
\end{aligned} \tag{5}
$$

General Observer Quantities

For the Weyl metric (4), we choose a unit time like velocity vector as,

$$u^k = \frac{1}{\sqrt{2}}\left(l^k + n^k\right) = \left(0,0,0,\frac{1}{\sqrt{f}}\right). \tag{6}$$

A field of observers with this velocity is expansion-free, shear-free and rotation-free. Also, the non-zero components of acceleration vector are,

$$a_1 = -\frac{f'}{2f}, \quad a_2 = -\frac{f_z}{2f}, \tag{7}$$

where prime (') denotes partial derivative with respect to r, and partial derivative with respect to z is denoted by suffix. The non-zero independent components of electric part of the Weyl tensor are,

$$
\begin{aligned}
E_{11} &= \frac{1}{6rf}\big(-3rf_z\gamma_z + rf_{zz} + rf\gamma_{zz} + f' \\
&\qquad - 3f\gamma' + 3rf'\gamma' - 2rf'' + rf\gamma''\big),\\
E_{12} = E_{21} &= \frac{1}{2rf}\left(-f\gamma_z + r\gamma_z f' + rf_z\gamma' + rf_z{}'\right),\\
E_{22} &= \frac{1}{6rf}\big(3rf_z\gamma_z - 2rf_{zz} + rf\gamma_{zz} + f' + 3f\gamma' \\
&\qquad - 3rf'\gamma' + rf'' + rf\gamma''\big),\\
E_{33} &= \frac{re^{-2\gamma}}{6f}\left(-2f'' + rf_{zz} + rf'' - 2rf\gamma_{zz} - 2rf\gamma''\right)
\end{aligned} \tag{8}
$$

and all components of magnetic parts of the Weyl tensor vanish. Thus, the Weyl metric is purely electric.

Newman-Penrose Quantities

The following are non-vanishing spin coefficients,

$$
\begin{aligned}
\kappa = -\nu &= \frac{e^{-\gamma}}{2\sqrt{2f}}\left(f\gamma_z - f_z\right),\\
\tau = -\pi &= -\frac{\sqrt{f}\,e^{-\gamma}}{2\sqrt{2}}\gamma_z,\\
\sigma = \lambda &= \frac{\sqrt{f}\,e^{-\gamma}}{2\sqrt{2}\,r}\left(1 - r\gamma'\right),\\
\rho = \mu &= \frac{e^{-\gamma}}{2\sqrt{2f}\,r}\left(rf' - f - rf\gamma'\right),\\
\varepsilon = \gamma &= \frac{e^{-\gamma}}{4\sqrt{2f}}\, f',\\
\alpha = -\beta &= \frac{e^{-\gamma}}{4\sqrt{2f}}\, f_z
\end{aligned} \tag{9}
$$

and NP Weyl scalars are as follows,

$$
\begin{aligned}
\Psi_0 = \Psi_4 &= -\frac{e^{-2\gamma}}{4r}\big(rf_z\gamma_z - rf_{zz} + rf\gamma_{zz} \\
&\qquad + f' + f\gamma' - rf'\gamma' + rf\gamma''\big),\\
\Psi_1 = \Psi_3 &= \frac{e^{-2\gamma}}{4r}\left(-f\gamma_z + rf'\gamma_z + rf_z\gamma' - rf_z{}'\right),\\
\Psi_2 &= -\frac{e^{-2\gamma}}{12r}\big(-3rf_z\gamma_z + rf_{zz} + rf\gamma_{zz} + f' \\
&\qquad 3f\gamma' + 3rf'\gamma' - 2rf'' + rf\gamma''\big).
\end{aligned} \tag{10}
$$

The following are non-vanishing Newman-Penrose complex scalars for Weyl metric,

$$
\begin{aligned}
\Phi_{00} = \Phi_{22} &= \frac{e^{-2\gamma}}{8rf}\big(-2rf_z{}^2 - 3rf'^2 + 2ff' \\
&\qquad + 2ff_{zz} + 2rff'' + 2f^2\gamma' \\
&\qquad - 2rf^2\gamma_{zz} - 2rf^2\gamma''\big),\\
\Phi_{01} = \Phi_{10} &= \frac{e^{-2\gamma}}{8rf}\left(2f^2\gamma_z - rf_z f'\right),\\
\Phi_{02} = \Phi_{20} &= \frac{e^{-2\gamma}}{8rf}\big(-rf_z{}^2 - 2f^2\gamma' - 2rf^2\gamma'' \\
&\qquad - 2rf^2\gamma_{zz}\big),
\end{aligned}
$$

$$\Phi_{11} = \frac{e^{-2\gamma}}{16rf}\left(-3rf_z^{\,2} - rf'^2 - 4f^2\gamma' + 2ff' + 2rff_{zz} + 2rff''\right),$$

$$\Phi_{12} = \Phi_{21} = \frac{e^{-2\gamma}}{8rf}\left(-2f^2\gamma_z + rf_z f'\right). \tag{11}$$

Thus, the metric (4) is non-vacuum and none of the Petrov types.

LANCZOS POTENTIAL

For Weyl metric (4) with tetrad (6), the field of observer u^k is shear-free and irrotational. Thus, the Lanczos potential [20] is given by,

$$L_{ijk} = a_i u_j u_k - a_j u_i u_k, \tag{12}$$

up to a gauge. To exhibit L_{ijk} in the Lanczos gauge, we have considered,

$$L_{ijk} = a_i u_j u_k - a_j u_i u_k - \frac{1}{3}\left(a_i g_{jk -} a_j g_{ik}\right). \tag{13}$$

The non-zero independent components of the Lanczos potential tensor for Weyl metric are as follow,

$$L_{121} = \frac{e^{2\gamma} f_z}{6f^2}, \qquad L_{122} = -\frac{e^{2\gamma} f'}{6f^2},$$

$$L_{133} = -\frac{r^2 f'}{6f^2}, \qquad L_{144} = -\frac{f'}{3}, \tag{14}$$

$$L_{233} = -\frac{r^2 f_z}{6f^2}, \qquad L_{244} = -\frac{f_z}{3}$$

and non-zero Lanczos potential scalars [16] as follows

$$L_0 = -3L_2 = 3L_5 = -L_7 = \frac{e^{-\gamma} f_z}{4\sqrt{2f}},$$

$$L_1 = L_6 = -\frac{e^{-\gamma} f'}{6\sqrt{2f}}. \tag{15}$$

RESULTS AND DISCUSSION

Using Lanczos scalars (15) and spin coefficients (9), it is possible to establish a linear relationship between them for the Weyl metric as,

$$L_0 = -3L_2 = 3L_5 = -L_7 = \alpha,$$

$$L_0 = L_6 = -\frac{2}{3}\varepsilon. \tag{16}$$

CONCLUSION

The Weyl metric is non-vacuum spacetime which is none of the Petrov types. Lanczos potential for the Weyl metric has been obtained. The Lanczos potential scalars depend on only two spin coefficients α and ε. Also, it supports the conjecture that there is linear relationship between Lanczos scalars and spin coefficients. It is hoped that the results obtained here will help researchers for finding the Lanczos potential for other non-vacuum spacetimes.

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REFERENCES

[1] Ahsan, Z., and Bilal, M. (2010): A Solution of Weyl-Lanczos Equations for Arbitrary Petrov Type D Vacuum Spacetimes. *International Journal of Theoretical Physics,* **49:** 2713-2722.

[2] Ahsan, Z., and Bilal, M. (2010): On the Lanczos Potential for Petrov Type N Spacetimes. *Journal of Vectorial Relativity,* **5:** 1-8.

[3] Ahsan, Z., and Bilal, M. (2011): Lanczos Potential and Perfect Fluid Spacetimes. *International Journal of Theoretical Physics,* **50:** 1752–1768.

[4] Ahsan, Z., and Bilal, M. (2012): On the Lanczos Potential for Petrov Type III Spacetimes. *Journal of Tensor Society,* **6:** 127-134.

[5] Ahsan, Z., and Bilal, M. (2013): Lanczos Potential for Arbitrary Petrov Type II Spacetimes. *International Journal of Theoretical Physics*, **52:** 4275-4282.

[6] Ares de Parga, G., Chayoya A., A., O., And Lopez Bonilla, J. L. (1989): Lanczos Potential. *Journal of Mathematical Physics,* **30:** 1294-1295.

[7] Caltenco, J., López-Bonilla, J., and Zúñiga-Segundo, A. (2002): Lanczos Spin tensor and GHP Formalism. *Czechoslovak Journal of Physics,* **52:** 901-909.

[8] Dolan, P., and Kim, C. W. (1994): Some Solutions of the Lanczos Vacuum Wave Equation. *Proceedings of the Royal Society* *of London A: Mathematical, Physical and Engineering Sciences*, **447:** 577–585.

[9] Dolan, P., and Muratori, B. D. (1998): The Lanczos Potential for Vacuum Space-Times with an Ernst Potential. *Journal of Mathematical Physics,* **39:** 5406-5420.

[10] Dolan, P., and Kim, C. W. (1994): Some Solutions of the Lanczos Vacuum Wave Equation. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences,* 447, **1931:** 577-585.

[11] Dolan, P., and Kim, C. W. (1994) The Wave Equation for the Lanczos Potential I. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences,* **447:** 557-575.

[12] Edgar, S. B., and Höglund, A. (2000): The Lanczos Potential for Weyl-Candidate Tensors Exists Only in Four Dimensions. *General Relativity and Gravitation,* **32:** 2307–2318.

[13] Hasmani, A. (2010): Algebraic Computation of Newman-Penrose Scalars in General Relativity using Mathematica. *Journal of Science,* **1:** 82–83.

[14] Hasmani, A., and Andharia, P. (2011): Algebraic Computation of Spin Coefficients in Newman-Penrose Formalism using Mathematica. *Journal of Dynamical Systems and Geometric, Theories* **9:** 27–36.

[15] Hasmani, A. H., and Panchal, R. (2015): Algebraic Computations of General Observer Quantities using Mathematica. *Astrophysics and Space Science,* **359:** 1–5.

[16] Hasmani, A. H., and Panchal, R. (2016): Lanczos Potential for Some Non-Vacuum Spacetimes. *The European Physical Journal Plus*, **131:** 1–6.

[17] Lanczos, C. (1962): The Splitting of the Riemann Tensor. *Rev. Mod. Phys.,* **34:** 379-389.

[18] Maher, W. F., and Zund, J. D. (1968): A Spinor Approach to the Lanczos Spin Tensor. *Il Nuovo Cimento A* (1965-1970), **57:** 638–648.

[19] Newman, E., and Penrose, R. (1962): An Approach to Gravitational Radiation by a Method of Spin Coefficients. *Journal of Mathematical Physics,* **3:** 566-578.

[20] Novello, M., and Velloso, A. L. (1987): The Connection between General Observers and Lanczos Potential. *General Relativity and Gravitation,* **19:** 1251–1265.

[21] O'Donnell, P. (2004): Letter: A Solution of the Weyl-Lanczos Equations for the Schwarzschild Space-Time. *General Relativity and Gravitation,* **36:** 1415-1422.

[22] Roberts, M. D. (1995): The Physical Interpretation of the Lanczos Tensor. *Il Nuovo Cimento B,* (1971-1996), **110:** 1165–1176.

[23] Weyl, H. (1917): The theory of gravitation. *Annalen Phys,* **54:** 1-14.