*PRAJÑĀ* - Journal of Pure and Applied Sciences Vol. 21: 35 - 37 (2013)
ISSN 0975 - 2595

# FRACTIONAL VERTEX COVERING NUMBER AND REGULAR GRAPHS

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## ABSTRACT

**In this paper, it is proved that for any regular graph with n vertices, the fractional vertex covering number is $\frac{n}{2}$. The result shows that this number, unlike vertex covering number, does not depend on the number of edges of the graph. Further it is also shown that if a graph G is such that each component of G contains a Hamiltonian cycle then also its fractional vertex covering number is $\frac{n}{2}$.**

***Keywords:*** *regular graph, Hamiltonian cycle, vertex covering set, vertex covering number, fractional vertex covering function, fractional vertex covering number.*

## Introduction

The concept of fractional vertex covering number is a fractional version of vertex covering number. A fractional vertex covering function has its range contained in [0, 1]. The fractional vertex covering number of the graph G is less than or equal to the vertex covering number of the graph. The vertex covering number of the graph depends on the number of edges in the graph. Infect if some edges are added in the graph, the vertex covering number is likely to increase but it will not decrease. For example, the vertex covering number of a path with five vertices is 2 but the vertex covering number of the complete graph with five vertices is 4.

In this paper, it will be proved that the fractional vertex covering number of a regular graph with n vertices is $\frac{n}{2}$. Of course, the only exception is the graph with no edges. Thus this fractional vertex covering number does not depend on the number of edges. This result can be extended to those graphs which contain Hamiltonian cycles.

**Preliminaries and Notations:** If G is a graph then V(G) will denote the vertex set of the graph G and E(G) will denote the edge set of the graph G. If S is a set then $|S|$ will denote the cardinality of the set S. If e is an edge of G, u and v are end vertices of e then we write e = uv or e = vu.

If $f$ is a function from V(G) to [0, 1] then for any edge e = uv, we will write $f^*(uv) = f(u) + f(v)$. The weight of the function $f$: V(G) → [0, 1] is denoted as $w(f)$ and defined as $w(f) = \sum_{v \in V(G)} f(v)$.

A regular graph in which the degree of each vertex is k $(k \geq 1)$ is called a k – regular graph. In this paper, only finite, simple and undirected graphs are considered.

**Definition 1:** (vertex covering set) A subset S of V(G) is said to be a vertex covering set of G if every edge of G has at least one end point in S [4].

**Definition 2:** (minimum vertex covering set) A vertex covering set of the graph G having minimum cardinality is called a minimum vertex covering set.

**Definition 3:** (vertex covering number) The cardinality of a minimum vertex covering set is called the vertex covering number of the graph G and it is denoted as $\alpha_0(G)$ [4].

**Definition 4:** (vertex covering function) A function $f: V(G) \rightarrow \{0, 1\}$ is said to be a vertex covering function if $f^*(uv) = f(u) + f(v) \geq 1$ for every edge uv of the graph G.

Thus the characteristic function of a vertex covering set is a vertex covering function. Motivated by the above definition, [4] defined fractional vertex covering function.

**Definition 5:** (fractional vertex covering function) A function $f: V(G) \rightarrow [0, 1]$ is called a fractional vertex covering function if $f^*(uv) = f(u) + f(v) \geq 1$ for every edge uv of the graph G [2].

**Definition 6:** (minimum fractional vertex covering function) A fractional vertex covering function is said to be a minimum fractional vertex covering function if its weight is minimum among all fractional vertex covering functions defined on V(G).

**Definition 7:** (fractional vertex covering number) The weight of a minimum fractional vertex covering function is called fractional vertex covering number of the graph G. It is denoted as $\alpha_f(G)$.

**Theorem 8:** If G is a k – regular graph $(k \geq 1)$ with n vertices then the fractional vertex covering number of G is equal to $\frac{n}{2}$.

**Proof:** If $f$ is a fractional vertex covering function of G then, $m = \sum_{e \in E(G)} f^*(e) \geq \frac{nk}{2}$ (since $nk = 2|E(G)|$).

If we calculate this number m using vertices of the graph G then each vertex v contributes $k f(v)$ and thus

$$m = \sum_{v \in V(G)} k f(v) = k\, w(f).$$

Thus $k\, w(f) = m \geq \frac{nk}{2}$

Hence $w(f) \geq \frac{n}{2}$.

Thus the minimum $w(f)$ could be $\frac{n}{2}$. Infect this minimum weight $\frac{n}{2}$ is also achieved by a function which

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is also a fractional vertex covering function. It is defined as $f(v) = \frac{1}{2}$ for all $v \in V(G)$.

Thus $\alpha_f(G) = \frac{n}{2}$ ■

**Remark 9:** By the above theorem, if $n \geq 2$ then the fractional vertex covering number of a complete graph $K_n$ is $\frac{n}{2}$. Also for the Petersen graph which has 10 vertices and which is 3 – regular, the fractional vertex covering number is 5. It may be noted that the vertex covering number of the Petersen graph is 6 and the vertex covering number of the complete graph $K_{10}$ is 9.

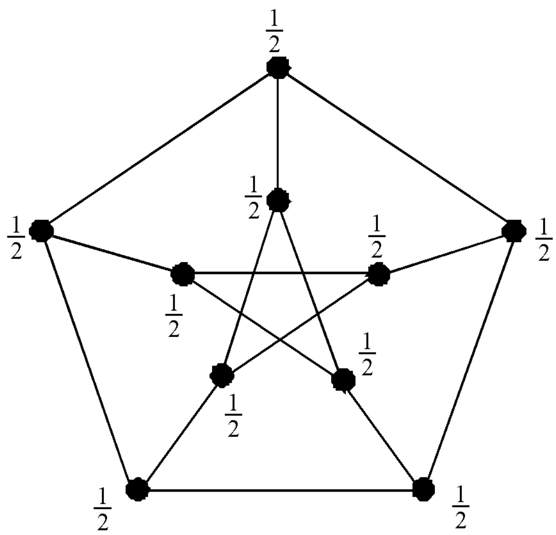


**Figure 1:** Petersen Graph

**Lemma 10:** Suppose that $H_1$ and $H_2$ are two regular graphs with $V(H_1) = V(H_2)$. Also suppose that H is a graph such that $V(H) = V(H_1)$ with $E(H_1) \subseteq E(H) \subseteq E(H_2)$ then $\alpha_f(H) = \frac{n}{2}$, where n is the number of vertices in H.

**Proof:** It is clear that $\alpha_f(H_1) \leq \alpha_f(H) \leq \alpha_f(H_2)$.

Now $\alpha_f(H_1) = \alpha_f(H_2) = \frac{n}{2}$.

Hence $\alpha_f(H) = \frac{n}{2}$. ■

**Corollary 11:** Let G be a graph which contains a Hamiltonian cycle and has n vertices then $\alpha_f(G) = \frac{n}{2}$.

**Proof:** Let $C_n$ denotes the Hamiltonian cycle of G, $H_1$ denotes the subgraph of $C_n$ and $H_2$ denotes the complete graph whose vertex set is V(G).

Now $\alpha_f(H_1) = \frac{n}{2}$ and $\alpha_f(H_2) = \frac{n}{2}$.

Therefore by Lemma 10, $\alpha_f(G) = \frac{n}{2}$. ■

**Definition 12:** (Cartesian product) The Cartesian product of two graphs G and H is denoted by G $\square$ H. For this graph $V(G \square H) = V(G) \times V(H)$ and $((u,u'), (v,v')) \in E(G \square H)$ if and only if $u = v$ and $(u',v') \in E(H)$ or $u' = v'$ and $(u,v) \in E(G)$ [3].

The Cartesian product $P_m \square P_n$ of two path graphs $P_m$ and $P_n$ is called a grid graph and it is denoted as $P_{mn}$.

Grid graphs have occupied an important place in graph theory due to its applications in engineering and computer science. In the following example, we provide the fractional vertex covering number of a grid graph.

**Example 13:** For m, n $\geq$ 1, consider the grid graph $P_{mn}$ then

(a) If m or n is even then $P_{mn}$ has Hamiltonian cycle and therefore the fractional vertex covering number of $P_{mn}$ is equal to $\frac{mn}{2}$.

(b) If m and n both are odd then the fractional vertex covering number of $P_{mn}$ is equal to $\frac{mn-1}{2}$.

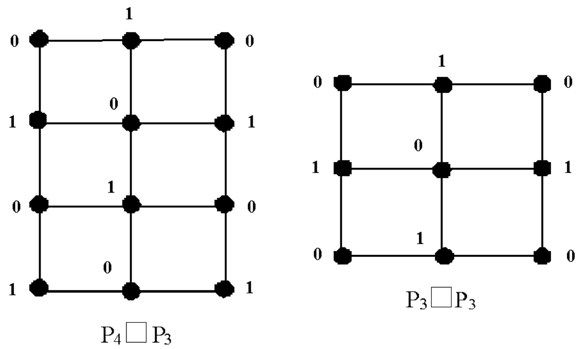


**Figure 2 :** Grid Graphs

The above example also shows that the fractional vertex covering number of a graph with n vertices may be less then $\frac{n}{2}$. The Cartesian product of two complete graphs has also been of great interest in graph theory.

**Remark 14:** If we consider two complete graphs $K_m$ and $K_n$ then their Cartesian product $K_m \square K_n$ is regular (with each vertex having degree m + n – 2). Therefore by Theorem 8, the fractional vertex covering number of $K_m \square K_n$ is $\frac{mn}{2}$.

We prove the following result on bipartite graph.

**Theorem 15:** If G is a bipartite graph with n vertices then

$$\alpha_f(G) \leq \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof:** Let $\{V_1, V_2\}$ be the bipartition of the vertex set V of the graph G. Suppose that $|V_1| \leq |V_2|$. Now define a function $f$ from V(G) to [0, 1] as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in V_1 \\ 0, & \text{if } x \in V_2 \end{cases}$$

Then obviously $f$ is a fractional vertex covering function of G. Therefore fractional vertex covering number of G is $\alpha_f(G) \leq$ weight of a function $f = |V_1| \leq \frac{n}{2}$.

Suppose that n is odd with $|V_1| < |V_2|$, then $|V_1| \leq \frac{n-1}{2}$.

Thus $\alpha_f(G) \leq \frac{n-1}{2}$. ■

**Corollary 16:** If T is a tree graph with n vertices then

$$\alpha_f(T) \leq \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

**Corollary 17:** If P is a path graph with n vertices then

$$\alpha_f(P) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

The fractional vertex covering number of a star graph with n vertices is much less than $\frac{n}{2}$.

**Example 18:** Consider any star graph with n vertices then its fractional vertex covering number is 1.

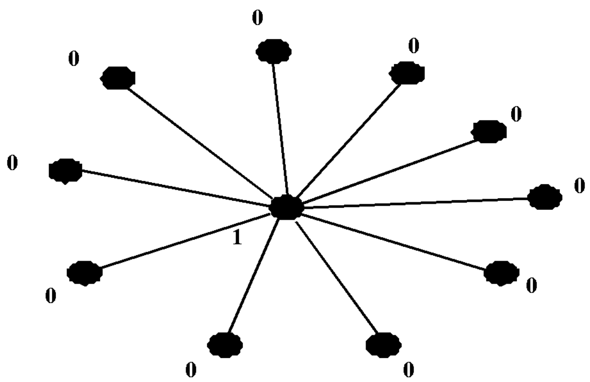


**Figure 3:** Star Graph

It is known that the fractional domination number of a tree graph is an integer [1]. It seems that the fractional vertex covering number of a tree graph should be an integer. We have the following:

**Open Problem:** Is the fractional vertex covering number of a tree graph necessarily an integer ?

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