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ERROR FUNCTIONS AND THEIR APPLICATION

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ABSTRACT

In this paper, authors studied origin of error function, its series form, its relationship with Confluent Hypergeometric function, incomplete Gamma functions. Laplace transform of error function and its generalization have been discussed. Differentiation and integration of error function and an engineering application of error function also been obtained.

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Introduction and Preliminaries

Error function is used in measurement theory (using probability and statistics), and although its use in other branches of mathematics has nothing to do with the characterization of measurement errors, the name has stuck. The Error function is a function which occurs in probability, statistics and partial differential equations.

Error function plays an important role in the theory of the normal random variable and probability determination.

In the present paper we compute the nth derivative of the error function in terms of Hermite function by using Power series representation of the error function, which is also used to discuss the relation of error function with confluent Hypergeometric function and incomplete gamma function. We compute the Laplace transform of the error function and suggest a generalization of the error function. We solved one dimensional Initial boundary Value problem for one dimensional diffusion equation in terms of error function.

Gamma function (Rainville[1]) is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt, \quad Re(\alpha) > 0 \qquad (1)$$

Incomplete Gamma function (Rainville[1]) of first kind is

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt \qquad (2)$$

Incomplete Gamma function (Rainville[1]) of second kind defined as

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt \qquad (3)$$

Error function (Rainville[1]) defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(-t^2) dt, \quad \text{x is a real} \qquad (4)$$

and complementary error function (Rainville[1]) defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty exp(-t^2) dt, \quad \text{x is a real} \qquad (5)$$

A Hypergeometric function pFq (Rainville[1]) defined as

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots \ldots a_p; \\ b_1, b_2, \ldots b_q; \end{matrix} \; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_n}{\prod_{m=1}^{q} (b_m)_n} \frac{z^n}{n!} \qquad (6)$$

where no denominator parameter bq is allowed to be zero or a negative integer.

Kummer's Theorem (Rainville[1]) states that b_q is neither zero nor negative integer then

$${}_1F_1(a; b; x) = e^x \, {}_1F_1(b-a; b; -x) \qquad (7)$$

Laplace transform (Rainville[1]) of f(t) defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt, \quad Re(s) > 0 \qquad (8)$$

Fourier transform (Lokenath Debnath[2]) of f(x) is denoted by

$\mathcal{F}\{f(x)\} = F(k), \; k \in \mathbb{R}$ and defined by the integral

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \qquad (9)$$

Inverse Fourier transform (Lokenath Debnath[2]) of f(x) is denoted by

$\mathcal{F}^{-1}\{F(k)\} = f(x)$ is defined by

$$\mathcal{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k) dk \qquad (10)$$

Fourier sine transform (Lokenath Debnath[2]) defined by

$$\mathcal{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, f(x) dx$$

Hermite polynomials Hn(x) and its recurrence relations (Rainville[1]) define as

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!} \qquad (11)$$

$$H_n'(x) = 2nH_{n-1}(x) \qquad (12)$$

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$$H_n(x) = 2x\, H_{n-1}(x) - 2(n-1)\, H_{n-2}(x) \quad (13)$$

Rodrigues' formula for Hermite polynomials (Rainville[1]) given by

$$H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2)$$

The Error Function

The Normal distribution was first introduced by de Moivre in 1733 in the context of approximating certain Binomial distributions for large n. His result was extended by Laplace in his book *Analytical Theory of Probabilities* (1812) and is now called the Theorem of de Moivre-Laplace. Laplace used the normal distribution in the analysis of errors of statistical distribution. The important numerical method of least squares was introduced by Legendre in 1805. Well-known Normal distribution define as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}; \quad (14)$$

$$-\infty < x < \infty, \sigma > 0$$

Considering $\mu = 0$ and $\sigma = 1$, the standard normal distribution defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\}; -\infty < x < \infty \quad (15)$$

Gauss considered the integral

$$I = \int \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\} du \quad (16)$$

Above integral can be written as

$$I = \frac{1}{\sqrt{\pi}} \int \exp(-t^2) dt \quad (17)$$

Gauss studied this integral in the range –x to x and he found that this integral is the error of normal distribution in terms of function as

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} \exp(-t^2) dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (18)$$

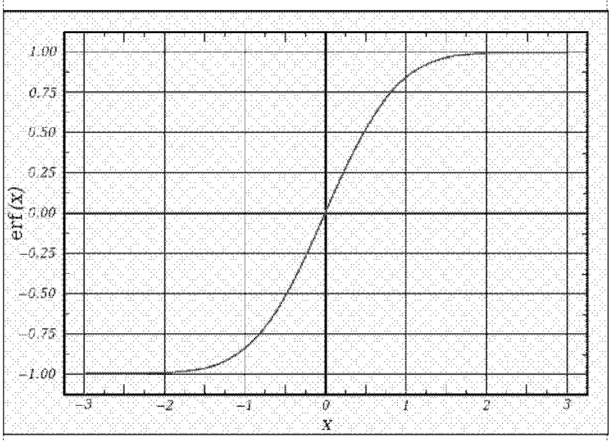

Figure 1: Graph of error function

By considering integral (17) in the range $-\infty$ to ∞ and using definition of Gamma function, we get

$$\int_{-\infty}^{\infty} \exp(-t^2) dt = \sqrt{\pi}$$

i.e. $\int_0^{\infty} \exp(-t^2) dt = \frac{\sqrt{\pi}}{2}$

This can be written

as $\frac{2}{\sqrt{\pi}} \left(\int_0^x \exp(-t^2) dt + \int_x^{\infty} \exp(-t^2) dt \right) = 1,$

this gives the definition of complementary error function as

$$erf_c(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2) dt$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (19)$$

$$= 1 - erf(x)$$

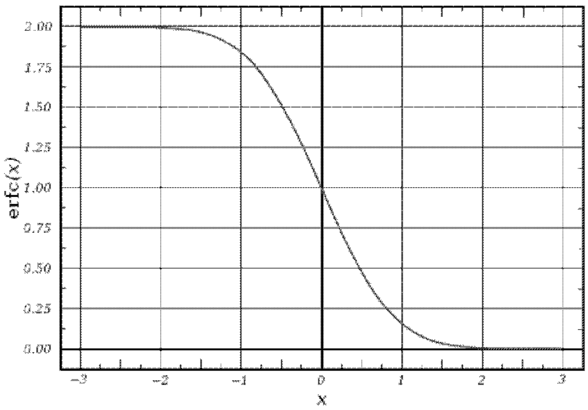

Figure 2: Graph of complementary error function

Elementary Properties of error function

1. $erf(\infty) = 1$
2. $erf(0) = 0$
3. $erf(-\infty) = -1$
4. $erfc(0) = 1$
5. $erfc(\infty) = 0$
6. $erf(-x) = -erf(x)$
7. $erfc(-x) = 1 + erf(x)$
8. $erfc(-x) + erfc(x) = 2$

Series Form of Error Function

Since the series for e^{-t^2} converges absolutely and uniformly for all $t \in R$, we have

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$
$$= \frac{2}{\sqrt{\pi}}\int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}dt \qquad (20)$$
$$= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}.$$

In view of the convergence behavior of the series, we further have term by term differentiation of (20) with respect to x, we get

$$\frac{d}{dx}(erf(x)) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(2n+1)x^{2n}}{2n+1}$$
$$= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} \qquad (21)$$
$$= \frac{2}{\sqrt{\pi}}e^{-x^2}$$

Second differentiation

$$\frac{d^2}{dx^2}(erf(x)) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}(e^{-x^2})$$

gives $= -\frac{4x}{\sqrt{\pi}}e^{-x^2}$ (22)

$$= \frac{2}{\sqrt{\pi}}(-2xe^{-x^2})$$

Third differentiation gives

$$\frac{d^3}{dx^3}(erf(x)) = -\frac{4}{\sqrt{\pi}}(1-2x^2)e^{-x^2}$$
$$= \frac{2}{\sqrt{\pi}}(4x^2-2)e^{-x^2} \qquad (23)$$

Now, we will derive following expression for n[th] derivative of error function by using mathematical induction on n,

$$\frac{d^n}{dx^n}(erf(x)) = (-1)^n\frac{2}{\sqrt{\pi}}H_{n-1}(x)e^{-x^2} \qquad (24)$$

where H_n is the Hermite polynomial

From equations (21) to (23), result (24) is true for n=1, 2, 3. Assume that (24) holds for n=k. So by induction hypothesis and using (12) and (13), we get

$$\frac{d^{k+1}}{dx^{k+1}}(erf(x)) = \frac{d}{dx}\left[\frac{d^k}{dx^k}(erf(x))\right]$$
$$= (-1)^k\frac{2}{\sqrt{\pi}}\frac{d}{dx}[H_{k-1}(x)e^{-x^2}]$$
$$= (-1)^k\frac{2}{\sqrt{\pi}}e^{-x^2}\left[2(k-1)H_{k-2}(x) - 2xH_{k-1}(x)\right]$$
$$= (-1)^{k+1}\frac{2}{\sqrt{\pi}}e^{-x^2}\left[2xH_{k-1}(x) - 2(k-1)H_{k-2}(x)\right]$$
$$= (-1)^{k+1}\frac{2}{\sqrt{\pi}}H_k(x)e^{-x^2}$$

Therefore, by mathematical induction equation (24) holds. Using integration by parts, we get following

$$\int_0^x erf(t)dt = x\,erf(x) + \frac{1}{\sqrt{\pi}}(e^{-x^2}-1), \quad x>0 \qquad (25)$$

$$\int_0^x erfc(t)dt = x\,erfc(x) + \frac{1}{\sqrt{\pi}}(1-e^{-x^2}), \quad x>0 \qquad (26)$$

$$\int_0^\infty erfc(t)\,dt = \frac{1}{\sqrt{\pi}} \qquad (27)$$

$$\int_a^b e^{-t^2}dt = \frac{\sqrt{\pi}}{2}(erf(b)-erf(a)), \quad b>a \qquad (28)$$

The Series form of an error function (Taylor's series at 0) is

$$erf(x) = \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5 - \frac{1}{21\sqrt{\pi}}x^7 + \frac{1}{108\sqrt{\pi}}x^9 - \ldots \qquad (29)$$

where the series in right hand side is absolutely and uniformly convergent. Hence

$$erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \ldots\right] \qquad (30)$$

Relationship With Confluent Hypergeometric Function

From equation (20), we get

$$erf(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}$$
$$= \frac{2x}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\frac{(-x^2)^n}{n!}$$
$$= \frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};-x^2\right) \qquad (31)$$

Using Kummer's theorem (7), we get

$$erf(x) = \frac{2x}{\sqrt{\pi}}e^{-x^2}\,{}_1F_1\left(1;\frac{3}{2};x^2\right) \qquad (32)$$

Relationship With Incomplete Gamma Functions

Incomplete Gamma function of first kind is

$$\gamma\left(\frac{1}{2},x^2\right) = \int_0^{x^2} e^{-t}t^{-\frac{1}{2}}dt = 2\int_0^x e^{-y^2}dy$$

Therefore,

$$\gamma\left(\frac{1}{2},x^2\right) = \sqrt{\pi}\;erf(x) \qquad (33)$$

The incomplete Gamma function of second kind is

$$\Gamma\left(\frac{1}{2}, x^2\right) = \int_{x^2}^{\infty} e^{-t} t^{-\frac{1}{2}} dt = 2\int_{x}^{\infty} e^{-y^2} dy$$

hence,

$$\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\ erfc(x) = \sqrt{\pi}(1 - erf(x)) \quad (34)$$

Laplace Transform of an Error Function

From definition of Laplace Transform

$$L[erf(t); s] = \int_0^{\infty} e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{t} e^{-u^2} du dt$$

Since e^{-t^2} is bounded and continuous for every t ($-\infty < t < \infty$), changing the order of integration we get

$$L[erf(t); s] = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \int_u^{\infty} e^{-st} dt du$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^{\infty} e^{-(u^2+su)} du$$

$$= \frac{2}{s\sqrt{\pi}} e^{\frac{s^2}{4}} \int_0^{\infty} e^{-(u+\frac{s}{2})^2} du$$

By taking u + s/2 = x, we get

$$L[erf(t); s] = \frac{2}{s\sqrt{\pi}} e^{\frac{s^2}{4}} \int_{\frac{s}{2}}^{\infty} e^{-x^2} dx$$

This reduces to,

$$L[erf(t); s] = \frac{1}{s} e^{\frac{s^2}{4}} erfc(\frac{s}{2}). \quad (35)$$

Similarly, the Laplace Transform of $erf(t^{1/2})$ is obtained as

$$L[erf(t^{\frac{1}{2}}); s] = \int_0^{\infty} e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{t^{\frac{1}{2}}} e^{-u^2} du dt$$

As e^{-t^2} is bounded and continuous for every t ($-\infty < t < \infty$), by changing the order of integration, we get

$$L[erf(t^{\frac{1}{2}}); s] = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} \int_{u^2}^{\infty} e^{-st} dt du$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^{\infty} e^{-(u^2+su^2)} du$$

$$= \frac{2}{s\sqrt{\pi}} \int_0^{\infty} e^{-(1+s)u^2} du$$

Taking substitution $(1 + s) u^2 = x^2$, we have

$$du = \frac{dx}{\sqrt{1+s}} \text{ so,}$$

$$L[erf(t^{\frac{1}{2}}); s] = \frac{2}{s\sqrt{\pi}\sqrt{1+s}} \int_0^{\infty} e^{-x^2} dx$$

$$= \frac{1}{s\sqrt{1+s}} \left(\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} dx \right)$$

Hence,

$$L[erf(t^{\frac{1}{2}}); s] = \frac{1}{s\sqrt{1+s}} \quad (36)$$

A Generalization

A generalization of error function (4) is

$$E_n(x) = \frac{n!}{\sqrt{\pi}} \int_0^{x} e^{-t^n} dt \quad (37)$$

where n = 0,1,2,….. The following results are immediately follows from (37),

$$E_0(x) = \frac{2}{\sqrt{\pi}} \left(\frac{x}{2e} \right)$$

$$E_1(x) = \frac{1}{\sqrt{\pi}} \int_0^{x} e^{-t} dt = \frac{2}{\sqrt{\pi}} \left[\frac{1-e^{x}}{2} \right]$$

$E_2(x) = erf(x)$

$$E_n(x) = \frac{n!}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{nk+1}}{(nk+1)!k!}$$

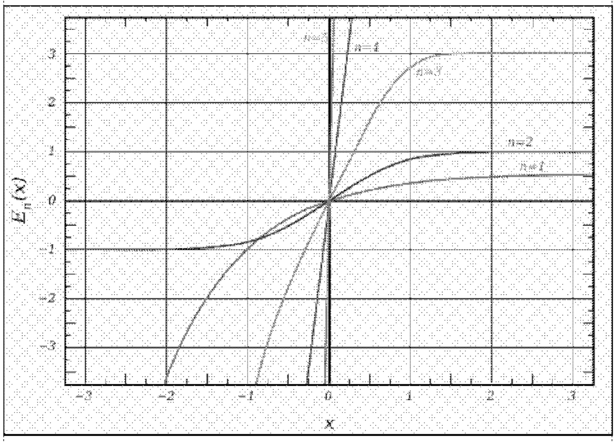

Figure 3: Graph of $E_n(x)$ case

An Application

Consider, the initial-boundary value problem for the one-dimensional diffusion equation with no sources or sinks

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \ -\infty < x < \infty, \ t > 0, \quad (38)$$

where is a diffusivity constant, with the initial condition

$$u(x, 0) = f(x), \ -\infty < x < \infty. \quad (39)$$

This problem can be transform in the following system

$$U_t = -\kappa k^2 U, \ t > 0,$$

$$U(k, 0) = F(k)$$

The solution of the above system is

$$U(k, t) = F(k) e^{-\kappa k^2 t}$$

The inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) exp[(ikx - \kappa k^2 t)] dk \quad (41)$$

Applying Convolution theorem,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x - \xi) d\xi \quad (42)$$

where,

$$g(x) = \Im^{-1}\{e^{-\kappa k^2 t}\} = \frac{1}{\sqrt{2\kappa t}} exp(-\frac{x^2}{4\kappa t}) \quad (43)$$

Therefore, solution (42) reduces to

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\kappa t} \right] d\xi \quad (44)$$

Using change of variable

$$\frac{\xi - x}{2\sqrt{\kappa t}} = \zeta, \quad d\zeta = \frac{d\xi}{2\sqrt{\kappa t}}$$

to express solution (44) in the form

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\sqrt{\kappa t}\zeta) \exp(-\zeta^2) d\zeta \quad (45)$$

This integral is convergent for all time $t > 0$ and the integrals obtained from (44) by differentiation under the integral sign with respect to x and t are uniformly convergent in the neighborhood of the point (x,t). Hence, the solution u(x,t) and its derivatives of all orders exist for $t > 0$.

We consider a special case involving discontinuous initial condition in the form $f(x) = T_0 H(x)$ where T_0 is constant. In this case solution (44) becomes

$$u(x,t) = \frac{T_o}{2\sqrt{\pi\kappa t}} \int_{0}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4\kappa t} \right] d\xi \quad (46)$$

By putting

$$\eta = \frac{\xi - x}{2\sqrt{\kappa t}},$$

solution (46) can be express in the form of error function as

$$u(x,t) = \frac{T_o}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{\kappa t}}}^{\infty} e^{-\eta^2} d\eta = \frac{T_o}{2} erf_c\left(-\frac{x}{2\sqrt{\kappa t}} \right)$$

$$= \frac{T_o}{2}\left[1 + erf\left(\frac{x}{2\sqrt{\kappa t}} \right) \right]$$

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