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A FIXED POINT THEOREM AND A COMMON FIXED POINT THEOREM

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ABSTRACT

The common fixed point theorem is a fruitful generalization of Banach's fixed point theorem. This theory has taken care of the convergence aspects also. Here it has been sought to extend and strengthen some of the well established variants of Banach's contraction principle. Besides, a theorem dealing with the convergence aspects of a variant of Banach fixed point theorem has been proved here.

Key Words: *Banach fixed point, common fixed point, uniqueness*

INTRODUCTION

The well known classical Banach Fixed Point Theorem reads as

Let (X, d) be a complete metric space and $T : X \to X$ satisfying $d(T(x), T(y)) \leq kd(x, y) \quad \forall x, y \in X$

and some k with $0 \leq k < 1$. Then T has has a unique fixed point in X.

This Theorem was generalized in certain direction by Browder and Petryshyn [1].

To start with the following result has been established which deals with the generalization of a variant of Banach fixed point theorem which is [5]

"Let X be a closed subset of a Hilbert space and $T : X \to X$ be a self mapping satisfying the following condition:

$$\|Tx - Ty\| \leq a_1 \frac{\|x - Ty\|\|Tx - x\|}{1 + \|x - y\|} + a_2 \frac{\|x - Ty\|[1 + \|Tx - x\|]}{1 + \|x - y\|} + a_3 \frac{\|Tx - x\|[1 + \|x - Ty\|]}{1 + \|x - y\|} + a_4\|x - y\|$$

for all $x, y \in X$ with $x \neq y$ where $a_i's$ are non negative real numbers with $0 \leq a_2 + a_3 + a_4 < 1$

Then T has a unique fixed point in X".

Theorem 1: Let X be a closed subeset of a Hilbert space and $T : X \to X$ be a self mapping satisfying the following condition:

$$\|Tx - Ty\|^2 \leq a_1 \frac{\|x - Ty\|^2\|Tx - x\|^2}{1 + \|x - y\|^2} + a_2 \frac{\|x - Ty\|^2[+\|Tx - x\|^2]}{1 + \|x - y\|^2} + a_3 \frac{\|Tx - x\|^2[+\|x - Ty\|^2]}{1 + \|x - y\|^2} + a_4\|x - y\|^2$$

for all $x, y \in X$ with $x \neq y$, where $a_i's$ are non negative real numbers with $0 \leq a_2 + a_3 + a_4 < 1$.

Then T has a unique fixed point in X.

Proof: Let x_0 be any arbitrary point in X. Let us define a sequence $\{x_n\}$ as follow:

$x_1 = Tx_0,$

$x_2 = Tx_1,$

....

$x_{n+1} = Tx_n,$

....

Now, we proceed to show that the sequence $\{x_n\}$ is a Cauchy sequence. For this observe that

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|$$

By the hypothesis one can observe that

$$\|x_{n+1} - x_n\|^2 \leq a_1 \frac{\|x_n - Tx_{n-1}\|^2\|Tx_n - x_n\|^2}{1 + \|x_n - x_{n-1}\|^2} + a_2 \frac{\|x_n - Tx_{n-1}\|^2[+\|Tx_n - x_n\|^2]}{1 + \|x_n - x_{n-1}\|^2} + a_3 \frac{\|Tx_n - x_n\|^2[+\|x_n - Tx_{n-1}\|^2]}{1 + \|x_n - x_{n-1}\|^2} + a_4\|x_n - x_{n-1}\|^2$$

which gives

$$\|x_{n+1} - x_n\|^2 \leq a_1 \frac{\|x_n - x_n\|^2\|x_{n+1} - x_n\|^2}{1 + \|x_n - x_{n-1}\|^2} + a_2 \frac{\|x_n - x_n\|^2[+\|x_{n+1} - x_n\|^2]}{1 + \|x_n - x_{n-1}\|^2} + a_3 \frac{\|x_{n+1} - x_n\|^2[+\|x_n - x_n\|^2]}{1 + \|x_n - x_{n-1}\|^2} + a_4\|x_n - x_{n-1}\|^2$$

implies

$$\|x_{n+1} - x_n\|^2 \leq a_3 \frac{\|x_{n+1} - x_n\|^2}{1 + \|x_n - x_{n-1}\|^2} + a_4\|x_n - x_{n-1}\|^2$$

leading to

$$\|x_{n+1} - x_n\|^2 + \|x_n - x_{n-1}\|^2\|x_{n+1} - x_n\|^2 \leq a_3\|x_{n+1} - x_n\|^2 + a_4\|x_n - x_{n-1}\|^2 + a_4\|x_n - x_{n-1}\|^4$$

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After simplifying this one finds that

$$\|x_{n+1}-x_n\|^2 \le p(n)\|x_n-x_{n-1}\|^2$$

where

$$p(n)=\frac{\left[a_4+a_4\|x_n-x_{n-1}\|^2\right]}{\left[\|x_n-x_{n-1}\|^2+1-a_3\right]}<1, \quad \forall n\ge 1$$

because

$a_2+a_3+a_4<1,$

one gets

$a_4\,|x_n-x_{n-1}|^2<|x_n-x_{n-1}|^2$

gives that

$a_4+a_4\,|x_n-x_{n-1}|^2<|x_n-x_{n-1}|^2+1-a_3$.

Continuing in this way, we find some $s<1$ such that $\|x_{n+1}-x_n\|^2\le s^n\|x_1-x_0\|^2$ for all $n\ge 1$

Consequently,

$\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$

Hence, $\{x_n\}$ is a Cauchy sequence in X, so by the completeness of X, there exists $\mu\in X$ with $\lim_{n\to\infty} x_n=\mu$

Also, $\{x_{n+1}\}=\{Tx_n\}$ is a subsequence of $\{x_n\}$ and hence converges to the same limit μ.

Since, T is continuous, one arrives at $T(\mu)=\mu$

That is, μ is a fixed point of T.

Next, to show the uniqueness of the fixed point let us take γ $(\gamma\ne\mu)$ to be another fixed point of T

that is, $T\gamma=\gamma$

and $\|\mu-\gamma\|\ne 0$

Now, it follows that

$$\|\mu-\gamma\|^2=\|T\mu-T\gamma\|^2$$

$$\le a_1\frac{\|\mu-\gamma\|^2\|\mu-\mu\|^2}{1+\|\mu-\gamma\|^2}+a_2\frac{\|\mu-\gamma\|^2\left[1+\|\mu-\mu\|^2\right]}{1+\|\mu-\gamma\|^2}$$

$$+a_3\frac{\|\mu-\mu\|^2\left[1+\|\mu-\gamma\|^2\right]}{1+\|\mu-\gamma\|^2}+a_4\|\mu-\gamma\|^2$$

$$\le a_2\frac{\|\mu-\gamma\|^2}{1+\|\mu-\gamma\|^2}+a_4\|\mu-\gamma\|^2$$

which implies that

$$\|\mu-\gamma\|^2+\|\mu-\gamma\|^4\le a_2\|\mu-\gamma\|^2+a_4\|\mu-\gamma\|^2+a_4\|\mu-\gamma\|^4$$

leading to

$$\|\mu-\gamma\|^2\le\left[\frac{a_2+a_4+a_4\|\mu-\gamma\|^2}{1+\|\mu-\gamma\|^2}\right]\|\mu-\gamma\|^2$$

since

$$\frac{a_2+a_4+a_4\|\mu-\gamma\|^2}{1+\|\mu-\gamma\|^2}<1$$

one concludes that $\mu=\gamma$, because

$0\le a_2+a_3+a_4<1$

results in

$a_4\,|\mu-\gamma|^2<|\mu-\gamma|^2$

Observe that the convergence is much more improved here.

Definition: Common fixed point

Let H be a Hilbert space and S, T:$H\to H$ be two mappings. A point x is said to be a common fixed point of S and T if $T(x)=x\ S(x)$.

On common fixed point theorems for three mapping in Hilbert Spaces: Koparde and Waghmode [2] extended the result of Jungck [3] and Fisher [4] regarding the existence and uniqueness of common fixed points. Here, we proceed to extend and sharpen the main result of Koparde and Waghmode [2] by modifying their procedure.

The incoming result deals with common fixed point theorem, wherein the convergence aspect has been duly addressed.

Theorem due to Koparde and Waghmode: Let S and T be continuous mappings of a Hilbert space X into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into $SX\cdot TX$ which commutes with S and T and satisfies the inequality,

$$\|Ax-Ay\|\le\alpha\|Ax-Sx\|+\beta\|Ay-Ty\|+\gamma\|Sx-Ty\|$$

for all x, y in X; α,β,γ are non-negative reals with $0<\alpha+\beta+\gamma<1$. Indeed, S, T and A then, have a unique common fixed point.

Theorem 2: Let S and T be continuous mappings of a Hilbert space X into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into $SX\cdot TX$ which commutes with S and T and satisfies the inequality,

$$\|Ax-Ay\|^2\le\alpha\|Ax-Sx\|\|Sy-Ty\|$$
$$+\beta\|Ay-Ty\|\|Sx-Ty\|$$
$$+\gamma\|Sx-Ty\|\|Tx-Sy\|+\delta\|Tx-Ty\|^2$$

for all x, y in X; where $\alpha,\beta,\gamma,\delta$ are non-negative reals with $0<\alpha+\beta+\gamma+\delta<1$. Indeed, S, T and A then, have a unique common fixed point.

Proof: Firstly, we prove that the existence of such a mapping A is necessary. For this suppose, $Sz=z=Tz$ for some z in X.

Define a mapping A of X into X by $Ax = z$ for all x in X. Then clearly, A is continuous mapping of X into $SX \cap TX$.

Since, $Sx, Tx \in X$, for all $x \in X$ and $Ax = z$, for all $x \in X$, one gets

$$ASx = z,$$
$$SAx = Sz = z,$$
$$ATx = z,$$
$$TAx = Tz = z.$$

Hence, A commutes with S and T. Now, for any $\alpha, \beta, \gamma, \delta$ with $0 < \alpha + \beta + \gamma + \delta < 1$, it is noticed that

$$\begin{aligned}\|Ax - Ay\|^2 &\le \alpha \|Ax - Sx\|\|Sx - Ty\| \\ &+ \beta \|Ay - Ty\|\|Sx - Ty\| \\ &+ \gamma \|Sx - Ty\|\|Tx - Sy\| + \delta \|Tx - Ty\|^2\end{aligned}$$

This proves the existence of such a mapping A is necessary.

To prove the sufficiency, a sequence $\{x_n\}$ is constructed as follows. Let $x_0 \in X$ be an arbitrary point. Since $AX \subset SX$, we choose a point x_1 in X such that $Sx_1 = Ax_0$. Also, $AX \subset TX$ and hence we can choose $x_2 \in X$ such that $Tx_2 = Ax_1$.

Continuing in this way, one finds a sequence $\{x_n\}$ as follows:

$$Sx_{2n-1} = Ax_{2n-2},\ Tx_{2n} = Ax_{2n-1}\ \ n = 1,2,3,...$$

Next, proceed to show that $\{Ax_n\}$ is a Cauchy sequence.

For this one arrives at the inequality,

$$\begin{aligned}\|Ax_{2n+1} - Ax_{2n}\|^2 &\le \alpha \|Ax_{2n+1} - Ax_{2n}\|\|Ax_{2n} - Ax_{2n-1}\| \\ &+ \beta \|Ax_{2n} - Ax_{2n-1}\|\|Ax_{2n} - Ax_{2n-1}\| \\ &+ \gamma \|Ax_{2n} - Ax_{2n-1}\|\|Ax_{2n} - Ax_{2n-1}\| \\ &+ \delta \|Ax_{2n} - Ax_{2n-1}\|^2 \\ &\le \alpha \left[\frac{\|Ax_{2n+1} - Ax_{2n}\|^2}{2} + \frac{\|Ax_{2n} - Ax_{2n-1}\|^2}{2}\right] \\ &+ (\beta + \gamma + \delta)\|Ax_{2n} - Ax_{2n-1}\|^2\end{aligned}$$

which gives

$$\|Ax_{2n+1} - Ax_{2n}\|^2 \le \left(\frac{\alpha + 2\beta + 2\gamma + 2\delta}{2 - \alpha}\right)\|Ax_{2n} - Ax_{2n-1}\|^2$$

Similarly, one concludes that

$$\|Ax_{2n} - Ax_{2n-1}\|^2 \le \left(\frac{\alpha + 2\beta + 2\gamma + 2\delta}{2 - \alpha}\right)\|Ax_{2n-1} - Ax_{2n-2}\|^2$$

Since, $0 < \alpha + \beta + \gamma + \delta < 1$, it follows that

$$0 < \left(\frac{\alpha + 2\beta + 2\gamma + 2\delta}{2 - \alpha}\right) < 1$$

This results in

$$\|Ax_{n+1} - Ax_n\|^2 \le \lambda^n \|Ax_1 - Ax_0\|^2$$

for all n, where $\lambda = \frac{\alpha + 2\beta + 2\gamma + 2\delta}{2 - \alpha}$.

Now, it can be seen from a simple calculation that $\{Ax_n\}$ is a Cauchy sequence and so it has a limit z in X. Since sequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are sub sequences of $\{Axn\}$, they have the same limit z. As S and A are commuting mapping, it is easy to see that $Sz = Az$ Similarly, one gets $Tz = Az$

Now, it can be observed that

$$\begin{aligned}\|Az - AAz\|^2 &\le \alpha \|Az - Sz\|\|Sz - TAz\| \\ &+ \beta \|AAz - TAz\|\|Sz - TAz\| \\ &+ \gamma \|Sz - TAz\|\|Tz - SAz\| + \delta \|Tz - TAz\|^2\end{aligned}$$

Since, $Tz = Az = Sz$ one obtains $\|AAz - TAz\| = 0$

as A commutes with S and T.

Hence one concludes that

$$\|Az - AAz\|^2 \le (\alpha + \gamma + \delta)\|Az - AAz\|^2$$

Since $\alpha + \gamma + \delta < 1$, this gives $Az = AAz$

Finally, putting $Az = z_1$, one can have $Az_1 = z_1$, $Tz_1 = z_1$, and $Sz_1 = z_1$, which means z_1 is a fixed point of S, T and A.

Next, to show uniqueness of this common fixed point, let us suppose that z_2 is also a common fixed point of S, T and A other than z_1, then $Sz_2 = Tz_2 = Az_2 = z_2$.

Note that $\|z_1 - z_2\| > 0$.

Now, it follows that

$$\begin{aligned}\|z_1 - z_2\|^2 &\le \alpha \|Az_1 - Sz_1\|\|Sz_2 - Tz_2\| \\ &+ \beta \|Az_2 - Tz_2\|\|Sz_1 - Tz_1\| \\ &+ \gamma \|Sz_1 - Tz_2\|\|Tz_1 - Sz_2\| \\ &+ \delta \|Tz_1 - Tz_2\|^2\end{aligned}$$

which leads to

$$\|z_1 - z_2\|^2 \le (\gamma + \delta)\|z_1 - z_2\|^2$$

This yields $z_1 = z_2$.

A comparison of the form of the inequalities given here and proved in a theorem by Koparde-Waghmode makes self evident that the current result is significantly improved and the current result is more sharpened as the convergence remains faster.

Conclusion

The methods adopted in the proofs of fixed point theorems reveal that yet there are various directions in which the Banach's fixed point theorem can be refined and extended retaining the convergence. A close look at the results found in this work indicates that there exist at least a couple of directions to improve the rate of convergence as well.

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