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COMPLEMENTARY COLOURING OF GRAPHS

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ABSTRACT

In this paper we introduce new concepts called complementary colouring and complementary chromatic number of a graph. This colouring is not a proper colouring in general. We prove that the complementary chromatic number of a graph cannot exceed the chromatic number of the graph. We also prove a necessary and sufficient condition under which the complementary chromatic number is same as the chromatic number of the graph. We also find a new upper bound for the complementary chromatic number of a given graph.

Key words: *Complementary colouring, Complementary chromatic number, Complete k – partite graph.*

AMS Subject Classification: 05C15.

INTRODUCTION

A colouring of a graph G is an assignment of a colour to each vertex of the graph. A proper colouring of a graph G is a colouring in which adjacent vertices receive different colours. Mathematically a colouring of a graph G is a function f from V(G) to a set $\{1, 2,\ldots,n\}$(for some $n \geq 1$). This colouring is a proper colouring if whenever u and v are adjacent vertices of G then $f(u) \neq f(v)$. In this paper we will define complementary colouring of a graph. In general this colouring is not a proper colouring.

Definitions and Preliminaries

Definition 1 [1] An n – colouring of a graph G is a function $f : V(G) \rightarrow \{1, 2,\ldots,n\}$(for some $n \geq 1$). This is called a proper n – colouring if whenever u and v are adjacent then $f(u) \neq f(v)$. A graph G is said to be n – colourable if it admits n – colouring and it is called a proper n – colourable graph if it admits a proper n – colouring.

Definition 2 [1] The chromatic number of a graph G is the smallest value of n such that G admits a proper n – colouring. This number is denoted as chr(G).

We introduce a new concept called complementary colouring of a graph and its complementary chromatic number.

Definition 3 A colouring of a graph G is called a complementary colouring of G if whenever two vertices u and v have distinct colours then u and v are adjacent. An n – colouring which is also complementary is called complementary n – colouring.

Definition 4 Complementary chromatic number of a graph G is the largest integer k such that G admits a complementary k – colouring. This number is denoted as cchr(G).

Definition 5 [1] A graph G is said to be a k - partite graph if its vertex set V(G) can be partitioned into k independent sets $V_1, V_2,\ldots,V_k$ such that each edge joins a vertex of V_i with a vertex of V_j for $j \neq i$. A k – partite graph G is said to be a complete k – partite graph if for each i, each vertex of V_i is adjacent to each vertex of V_j for every $j \neq i$.

Every graph G admits a complementary colouring because we can assign colour 1 to each vertex of the graph G. So it is obvious that this cannot exceed n – the number of vertices of the graph G.

For any $n \geq 1$, K_n will denote the complete graph with n vertices, C_n will denote the cycle graph with n vertices, P_n will denote the path with n vertices and $\delta(G)$ denotes the minimum degree of graph G.

We will consider only simple graphs in this article. We will also assume that any n – colouring or proper n- colouring or complementary n – colouring is an onto function from V(G) to $\{1,2,\ldots,.n\}$.

Examples: $cchr(K_n) = n$, $cchr(P_3) = 2$, $cchr(P_n) = 1$, if $n \geq 4$, $cchr(C_n) = 1$, if $n \geq 5$and If G is a disconnected graph then $cchr(G) = 1$.

Any proper k - colouring of a graph gives rise to a partition of k maximal independent sets namely the colour classes of this colouring.

Now we prove that the complementary chromatic number of a graph cannot exceed the chromatic number of the graph.

Theorem 1: If G is a graph then $cchr(G) \leq chr(G)$.

Proof: Let k = chr(G). We will prove that if $j > k$ then G does not admit a complementary j - colouring. Suppose for some $j > k$ G admits a complementary j – colouring. Consider a proper k – colouring of G. For each i, let $C_i = \{v \in V(G)/ v$ get colour i in this proper k – colouring$\}$.(i = 1,2,…k). For each i, let $T_i = \{v \in V(G)/v$ gets colour i in the complementary j - colouring$\}$.(i = 1,2,…,j). The sets $C_1, C_2,\ldots C_k$ are maximal independent and their union is

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V(G). Note that k is the smallest positive integer such that V(G) admits a partition into k maximal independent sets. Since the vertices having the same colour in the proper k – colouring receive the same colour in the complementary j - colouring, we have each $C_i \subset T_{mi}$ for some $m_i \leq j$. Also note that $\{T_1, T_2, \ldots T_j\}$ is a partition of V(G) and $\{C_1, C_2, \ldots C_k\}$ is also a partition of V(G) it follows that for some r, T_r must be empty. This contradicts the fact that $\{T_1, T_2, \ldots T_j\}$ forms a partition of V(G). Thus there is no complementary j – colouring if j > k. Thus if G admits a complementary j - colouring then $j \leq k$. And hence cchr(G) ≤ chr(G).

Now we prove a necessary and sufficient condition under which the above two numbers are equal.

Theorem 2: For a graph G, cchr(G) = chr(G) = k iff G is a complete k – partite graph.

Proof: First suppose that G is a complete k – partite graph with a partition into k independent sets $V_1, V_2, \ldots V_k$. Assign colour i to all vertices of V_i (i = 1,2,...,k). Then this is a proper k – colouring as well as a complementary k – colouring. Also note that this colouring implies that the chromatic number of G is equal to k. Since this colouring is also a complementary k- colouring it implies that cchr(G) = k = chr(G). To prove converse suppose cchr(G) = chr(G) = k. Consider any proper k – colouring of G.We have already proved in previous theorem that for any j > k there is no complementary j – colouring of G. Suppose there is k – colouring of G which is complementary. Let T_i and C_i be defined as in the previous theorem(i = 1, 2,..., k). Again each $C_i \subset T_{mi}$ for some $m_i \leq k$.(i = 1,2,.,k). Since $\{C_1, C_2, \ldots C_k\}$ and $\{T_{m1}, T_{m2}, \ldots T_{mk}\}$ both are partitions of V(G), each $T_{mi} = C_i$. Now we prove that $\{T_{m1}, T_{m2}, \ldots T_{mk}\}$ makes G a complete k – partite graph. Since each $T_{mi} = C_i$, it is an independent set. Let x be any vertex in T_{mi} and y be any vertex in T_{mj} $(i \neq j)$. Since x and y have different colours they must be adjacent. So G is a complete k – partite graph.This completes the proof.

Theorem 3: If G is a graph which admits a complementary n – colouring then $n \leq \delta(G) + 1$.

Proof: Consider a complementary n – colouring of G. Suppose its colour classes are $C_1, C_2, \ldots, C_n$. For each i select exactly one vertex x_i from C_i. The subgraph H induced by the vertices $x_1, x_2, \ldots, x_n$ is complete. The colouring induced on H is proper and it uses all n colours. Moreover if y is any vertex of G which is not in H then y receives one of the n colours. And so this vertex must be adjacent to atleast n -1 vertices of H. Also every vertex of H is adjacent to atleast n – 1 vertices of G. Thus degree of every vertex of G is greater than or equal to n – 1. Hence $\delta(G) \geq n - 1$. Therefore $n \leq \delta(G) + 1$.

Corollary: If G is a graph then the cchr(G) ≤ min $\{chr(G), \delta(G) + 1\}$

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