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ON A NEW SEQUENCE OF FUNCTIONS DEFINED BY A GENERALIZED RODRIGUES FORMULA

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ABSTRACT

In present paper, an attempt is made to provide an elegant unification of several classes of polynomials. Introduced sequence of the functions $\{V_n^{(\alpha,\beta)}(x,a,k,s)$ /n = 0, 1, 2…} by means of generalized Rodrigues formula (7), which involves Mittag-Leffler function $E_\alpha(z)$ and other two similar kind of class of polynomials (5), (6). Some generating relations and finite summation formulae have also been obtained for (7).

Keywords: *Mittag-Leffler function, generating relations, finite summation formulae.*

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INTRODUCTION:

In 1971, Srivastava and Singhal [8] introduced general class of polynomial $G_n^{(\alpha)}(x,r,p,k)$ by employing the operator $x^{k+1}D$, defined as

$$G_n^{(\alpha)}(x,r,p,k) = 1/n!\; x^{-\alpha-kn} \exp(px^r)(x^{k+1}D)^n[x^\alpha \exp(-px^r)] \quad (1)$$

Mittal [2] proved a Rodrigues formula for a class of polynomials $T_{kn}^{(\alpha)}(x)$ is given by

$$T_{kn}^{(\alpha)}(x) = 1/n!\; x^{-\alpha} \exp\{p_k(x)\} \times D_x^n[x^{\alpha+n} \exp\{-p_k(x)\}] \quad (2)$$

where $p_k(x)$ is a polynomial in x of degree k. Mittal [2] also proved the following relation

$$T_{kn}^{(\alpha+s-1)}(x) = 1/n!\, x^{-\alpha-n} \exp\{p_k(x)\} \times (T_s)^n[x^\alpha \exp\{-p_k(x)\}] \quad (3)$$

where, $T_s \equiv x(s+xD)$.

In 1979, Srivastava and Singh [7] introduced a general sequence of functions $V_n^{(\alpha)}(x;a,k,s)$ by employing the operator $\theta \equiv x^a(s+xD)$ where a and s are constants, defined as

$$V_n^{(\alpha)}(x;a,k,s) = 1/n!\, x^{-\alpha} \exp\{p_k(x)\}\, \theta^n[x^\alpha \exp\{-p_k(x)\}] \quad (4)$$

where $p_k(x)$ is a polynomial in x of degree k.

The new sequences of functions introduced in this paper are defined by (5) and (6) in the generalized form of (1) and (2) respectively as:

$$G_n^{(\alpha,\beta)}(x,r,p,k) = 1/n!\, x^{-\beta-kn} E_\alpha(px^r)(x^{k+1}D)^n[x^\beta 1/E_\alpha(px^r)], \quad (5)$$

$$T_{kn}^{(\alpha,\beta+s-1)}(x) = 1/n!\, x^{-\beta-n} E_\alpha\{p_k(x)\} \times (T_s)^n[x^\beta 1/E_\alpha\{p_k(x)\}] \quad (6)$$

In this paper, authors also introduced one more new sequence of functions $\{V_n^{(\alpha,\beta)}(x;a,k,s)/n=0,1,2,...\}$ by means of generalized Rodrigues formula (7)

$$V_n^{(\alpha,\beta)}(x;a,k,s) = 1/n!\, x^{-\beta} E_\alpha\{p_k(x)\}\, \theta^n[x^\beta 1/E_\alpha\{p_k(x)\}] \quad (7)$$

where, $\theta \equiv x^a(s+xD)$; a, s are constants and $\alpha \geq 0$, α and β are real or complex numbers; $n = 0,1,2,...$, k is finite and non-negative integer, $p_k(x)$ is a polynomial in x of degree k with

$x \in (0,\infty)$

The Mittag-Leffler function [1] defined as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \quad (8)$$

where z is a complex variable and $\Gamma(s)$ is a gamma function, $\alpha \geq 0$. The Mittag–Leffler function is direct generalization of the hypergeometric function $\frac{1}{1-z}$ and exponential function e^z to which it reduces for $\alpha = 0$ and $\alpha = 1$, i.e. $E_0(z) = \frac{1}{1-z}$ and $E_1(z) = e^z$. Its importance is realized during the last two decades due to its direct involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag–Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations. Therefore, it is obvious that (1), (2) and (4) are special cases of (5), (6) and (7) respectively for $\alpha = 1$.

In the present paper, the generating relations and finite summation formulae obtained for sequence of functions (7) as these are obviously more powerful sequence of functions than (5) and (6). The technique discussed in this paper will certainly apply for sequence of functions (5) and (6).

To obtain generating relations and finite summation formulae, the properties of the differential operators $\theta \equiv x^a(s+xD)$ and $\theta_1 \equiv x^a(1+xD)$, where $D \equiv \frac{d}{dx}$ used on the based of work (Mittal [3], Patil and Thakare [5]).

In the fourth section, the relations between (7) with some well-known polynomials (9) and (10) also have been discussed.

Hermite polynomials (Rainville [6]) defined as:

$$H_n(x) = (-1)^n \exp(x^2) D^n[\exp(-x^2)] \quad (9)$$

Konhauser polynomials of first kind (Srivastava [9]) defined as:

$$Y_n^\alpha(x;k) = \frac{x^{-kn-\alpha-1}e^x}{k^n\, n!}(x^{k+1}D)^n[x^{\alpha+1}e^{-x}] \quad (10)$$

GENERATING RELATIONS:

We obtained some generating relations of (7) as,

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha,\beta)}(x;a,k,s)t^n = (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{E_\alpha[p_k(x)]}{E_\alpha[p_k\{x(1-at)^{-\frac{1}{a}}\}]} \quad (11)$$

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha,\beta-an)}(x;a,k,s)t^n = (1+at)^{\frac{\beta+s}{a}-1} \frac{E_\alpha[p_k(x)]}{E_\alpha[p_k\{x(1+at)^{\frac{1}{a}}\}]} \quad (12)$$

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$$\sum_{m=0}^{\infty} x^{-am}\binom{m+n}{n} V_{n+m}^{(\alpha,\beta)}(x;a,k,s)\,t^m = (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{E_\alpha\{p_k(x)\}}{E_\alpha\left[p_k\{x(1-at)^{-1/a}\}\right]}\times V_n^{(\alpha,\beta)}\{x(1-at)^{-1/a};a,k,s,\} \quad (13)$$

Proof of (11):

From (7), we consider

$$\sum_{n=0}^{\infty} V_n^{(\alpha,\beta)}(x;a,k,s)\,t^n = x^{-\beta} E_\alpha\{p_k(x)\}\, e^{t\theta} x^{\beta}\left[\frac{1}{E_\alpha\{p_k(x)\}}\right]$$

above equation reduces to

$$\sum_{n=0}^{\infty} V_n^{(\alpha,\beta)}(x;a,k,s)\,t^n = x^{-\beta} E_\alpha\{p_k(x)\}\, x^{\beta}\,(1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}\times \frac{1}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}$$

and replacing t by tx^{-a}, which gives (11).

Proof of (12):

From (7), we consider

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha,\,\beta-an)}(x;a,k,s)\,t^n = x^{-\beta} E_\alpha\{p_k(x)\}\, e^{t\theta}\left[x^{\beta-an}\frac{1}{E_\alpha\{p_k(x)\}}\right]$$

and simplifying the above equation, we get

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha,\,\beta-an)}(x;a,k,s)\,t^n = x^{-\beta} E_\alpha\{p_k(x)\}\, x^{\beta}(1+at)^{\frac{\beta+s}{a}-1}\times \frac{1}{E_\alpha[p_k\{x(1+at)^{1/a}\}]}$$

which proves (12).

Proof of (13):

writing (7) as

$$\theta^n[x^\beta E_\alpha\{-p_k(x)\}] = n!\,x^\beta \frac{1}{E_\alpha\{p_k(x)\}} V_n^{(\alpha,\beta)}(x;a,k,s)$$

or

$$e^{t\theta}\left(\theta^n[x^\beta E_\alpha\{-p_k(x)\}]\right) = n!\,e^{t\theta}\left[x^\beta \frac{1}{E_\alpha\{p_k(x)\}} V_n^{(\alpha,\beta)}(x;a,k,s)\right]$$

above equation can be written as

$$\sum_{m=0}^{\infty}\frac{t^m\theta^{m+n}}{m!}[x^\beta E_\alpha\{-p_k(x)\}] = n!\,x^\beta (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}\times \frac{1}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}\times V_n^{(\alpha,\beta)}\{x(1-ax^a t)^{-1/a};a,k,s,\}$$

$$= \sum_{m=0}^{\infty}\frac{1}{m!\,n!}(m+n)!\,x^\beta \frac{1}{E_\alpha\{p_k(x)\}}\times V_{m+n}^{(\alpha,\beta)}(x;a,k,s)\,t^m$$

$$= x^\beta (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)} \frac{1}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}\times V_n^{(\alpha,\beta)}\{x(1-ax^a t)^{-1/a};a,k,s,\}$$

and above expression reduces to

$$\sum_{m=0}^{\infty}\binom{m+n}{n} V_{m+n}^{(\alpha,\beta)}(x;a,k,s)\,t^m = (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)} \frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}\times V_n^{(\alpha,\beta)}\{x(1-ax^a t)^{-1/a};a,k,s,\}$$

replacing t by tx^{-a}, which leads to (13).

FINITE SUMMATION FORMULAE:

We obtained two finite sum formulae for (7) as

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \sum_{m=0}^{n}\frac{1}{m!}(ax^a)^m\left(\frac{\beta}{a}\right)_m V_{n-m}^{(\alpha,0)}(x;a,k,s) \quad (14)$$

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \sum_{m=0}^{n}\frac{1}{m!}(ax^a)^m\left(\frac{\beta-\gamma}{a}\right)_m V_{n-m}^{(\alpha,\gamma)}(x;a,k,s) \quad (15)$$

Proof of (14):

We can write (7) as,

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \frac{1}{n!}x^{-\beta} E_\alpha\{p_k(x)\}\,\theta^n\left[x\,x^{\beta-1}\frac{1}{E_\alpha\{p_k(x)\}}\right]$$

we get,

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \frac{1}{n!}x^{-\beta}E_\alpha\{p_k(x)\}\,x\times \sum_{m=0}^{n}\binom{n}{m}\theta^{n-m}\left[\frac{1}{E_\alpha\{p_k(x)\}}\right]\theta_1^m\left(x^{\beta-1}\right)$$

which yields

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \frac{1}{n!}x^{-\beta+1}E_\alpha\{p_k(x)\}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\times x^{a(n-m)}[(s+xD)(s+a+xD)(s+2a+xD)\ldots\ldots(s+(n-m-1)a+xD)]\left[\frac{1}{E_\alpha\{p_k(x)\}}\right]\times x^{am}[(1+xD)(1+a+xD)(1+2a+xD)\ldots\ldots(1+(m-1)a+xD)]\,x^{\beta-1}$$

$$= E_\alpha\{p_k(x)\}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}x^{an}\times \prod_{i=0}^{n-m-1}(s+ia+xD)\left[\frac{1}{E_\alpha\{p_k(x)\}}\right]a^m\left(\frac{\beta}{a}\right)_m \quad (16)$$

Putting $\beta = 0$ and replacing n by n-m in (7) then equation reduces to

$$V_{n-m}^{(\alpha,0)}(x;a,k,s) = \frac{1}{(n-m)!}E_\alpha\{p_k(x)\}\theta^{n-m}\left[\frac{1}{E_\alpha\{p_k(x)\}}\right]$$

thus, we have

$$\frac{1}{(n-m)!}\theta^{n-m}\left[\frac{1}{E_\alpha\{p_k(x)\}}\right] = \frac{1}{E_\alpha\{p_k(x)\}}V_{n-m}^{(\alpha,0)}(x;a,k,s)$$

we get,

$$\frac{1}{(n-m)!}\prod_{i=0}^{n-m-1}(s+ia+xD)\left[\frac{1}{E_\alpha\{p_k(x)\}}\right] = \frac{x^{a(m-n)}}{E_\alpha\{p_k(x)\}}V_{n-m}^{(\alpha,0)}(x;a,k,s) \quad (17)$$

use of (17) and (16), which immediately leads (14).

Proof of (15):

From (7) we consider,

$$\sum_{n=0}^{\infty}V_n^{(\alpha,\beta)}(x;a,k,s)\,t^n = x^{-\beta}E_\alpha\{p_k(x)\}\,e^{t\theta}\left[x^\beta\frac{1}{E_\alpha\{p_k(x)\}}\right]$$

above equation reduces to,

$$\sum_{n=0}^{\infty}V_n^{(\alpha,\beta)}(x;a,k,s)\,t^n = (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}\frac{E_\alpha[p_k(x)]}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}$$

$$= x^{-\beta}E_\alpha\{p_k(x)\}\,x^\beta\,(1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}\times \frac{1}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}$$

which yields

$$\sum_{n=0}^{\infty} V_n^{(\alpha,\beta)}(x;a,k,s)\,t^n = (1-ax^a t)^{-\left(\frac{\gamma+s}{a}\right)} \sum_{m=0}^{\infty}\left(\frac{\beta-\gamma}{a}\right)_m \frac{(ax^a t)^m}{m!} \times \frac{E_\alpha[p_k(x)]}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}$$

by using (11),

$$\sum_{n=0}^{\infty} V_n^{(\alpha,\beta)}(x;a,k,s)\,t^n = \sum_{m=0}^{\infty}\left(\frac{\beta-\gamma}{a}\right)_m \frac{(ax^a t)^m}{m!} x^{-\gamma} \times E_\alpha\{p_k(x)\}e^{t\theta}\left[x^\gamma \frac{1}{E_\alpha\{p_k(x)\}}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\beta-\gamma}{a}\right)_m \frac{(ax^a)^m t^{n+m}}{m!\,n!} x^{-\gamma} \times E_\alpha\{p_k(x)\}\theta^n\left[x^\gamma \frac{1}{E_\alpha\{p_k(x)\}}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\beta-\gamma}{a}\right)_m \frac{(ax^a)^m t^n}{m!\,(n-m)!} x^{-\gamma} \times E_\alpha\{p_k(x)\}\theta^{n-m}\left[x^\gamma \frac{1}{E_\alpha\{p_k(x)\}}\right]$$

by equating the coefficients of t^n, we get

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \sum_{m=0}^{n}\frac{1}{m!}(ax^a)^m\left(\frac{\beta-\gamma}{a}\right)_m \frac{x^{-\gamma}}{(n-m)!} \times E_\alpha\{p_k(x)\}\theta^{n-m}\left[x^\gamma \frac{1}{E_\alpha\{p_k(x)\}}\right]$$

and use of (7), we get (15).

SPECIAL CASES:

In this section, we have obtained some special cases and relations of sequence of functions $V_n^{(\alpha,\beta)}(x;a,k,s)$, in the connection of (4), (9) and (10): Putting $\alpha = 1$ and replacing β by α in (7) then

$$V_n^{(1,\alpha)}(x;a,k,s) = V_n^{(\alpha)}(x;a,k,s) \tag{18}$$

Therefore, we can say that (4) is a particular case of (7).

If $\alpha = 1$, replacing β by $\alpha+1$, $\alpha=1$, $p_k(x) = p_1(x) = x$, and $s=0$, then (7) reduces to

$$V_n^{(1,\alpha+1)}(x;1,1,0) = x^n Y_n^\alpha(x;1) \tag{19}$$

if $\alpha = 1$, $\beta = 0$, $p_k(x) = p_2(x) = x^2$, and $s=0$, then (7) reduces to $a=-1$ and $s=0$ then (7) gives

$$V_n^{(1,0)}(x;-1,2,0) = \frac{(-1)^n}{n!} H_n(x) \tag{20}$$

CONCLUSION:

The new sequence of functions (5), (6) and (7), introduced the in section 1, the results obtained in sections 2, 3 and 4 seems to be new and quite interesting.

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