SC

No of printed pages: **2**

SEAT No.________

[209]

Sardar Patel University
Mathematics
M.Sc. Semester III
Monday, 29 October 2018
2.00 p.m. to 5.00 p.m.
PS03EMTH40 - Problems and Exercises in Mathematics I

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $a, b, c \in \mathbb{R}$ and $b \neq 0$. Then $f(x) = ax^2 + bx + c\ (x \in \mathbb{R})$ has minimum if and only if

(a) $a > 0$ (b) $a < 0$ (c) $a = 0$ (d) $c = 0$

(2) The value of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is

(a) 0 (b) 1 (c) 2 (d) ∞

(3) The radius of convergence of the Taylor series of $\frac{1}{z^2+1}$ about -1 is

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $\sqrt{5}$

(4) The closure of $\{e^z : z \in \mathbb{C}\}$ in $\mathbb{C}$ is

(a) $\mathbb{C}$ (b) $\mathbb{C} \setminus \{0\}$ (c) $[0, \infty)$ (d) $\mathbb{R}$

(5) ________ is not a topological property

(a) Being Hausdorff (b) Being dense (c) Completeness (d) None of these

(6) Let $(X, \mathscr{T})$ be any topological space such that $X \setminus \{x\}$ is open for all $x \in X$. Then $(X, \mathscr{T})$ is ________.

(a) T_1 (b) T_2 (c) compact (d) discrete

(7) A group of order ________ need not be abelian.

(a) 25 (b) 31 (c) 35 (d) 55

(8) Let G be a finite group such that $7 \mid o(G)$. Then the number of 7-Sylow subgroups of G can possibly be ________.

(a) 0 (b) 21 (c) 29 (d) none of these

Q.2 Attempt any *Seven*. [14]

(a) Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $\alpha \in \mathbb{R}$ and if $f(a) = f(b) = 0$, then show that there is $c \in (a, b)$ such that $f'(c) + e^{\alpha c} f(c) = 0$.

(b) Let $\{a_n\}$ be a real sequence such that $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$. Show that $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-1, 1)$.

(c) Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{[\frac{n}{2}]} \cos(\frac{k\pi}{n})$.

(d) Let f be analytic in a domain D and $f(z_0) = 0$ for some $z_0 \in D$. Show that either $f \equiv 0$ or there is a deleted neighbourhood of z_0 on which f is nowhere zero.

(P.T.O.)

(e) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If $f : \mathbb{D} \to \mathbb{C}$ is analytic and $|f(z)| \leq 1 - |z|$ $(z \in \mathbb{D})$, then show that $f \equiv 0$.
(f) For non-empty sets A and B prove or disprove $A^\circ \cup B^\circ = (A \cup B)^\circ$, where A° denotes the interior of the set A.
(g) Is $\mathbb{R}$ with lower limit topology homeomorphic to $\mathbb{R}$ with upper limit topology? Justify.
(h) Let G be a finite group and ϕ be an automorphism of G. Show that $o(g) = o(\phi(g))$ for every $g \in G$.
(i) Let H be a subgroup of a group G with index 2 in G. Show that H is normal in G.

Q.3

(a) If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then show that $f(x^+)$ and $f(x^-)$ exist for all $x \in \mathbb{R}$. Deduce that f is continuous at x if and only if $f(x^-) = f(x^+)$. [6]

OR

(a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined as $f_n(x) = \frac{1}{1+(x-n)^2}$ for all $x \in \mathbb{R}$. Show that $\{f_n\}$ converges uniformly on $(-\infty, 0)$ but fails to converge uniformly on $(0, \infty)$. State the results you use. [6]
(b) If $f : [a, b] \to \mathbb{R}$ is a continuous function, then show that $\lim_{n\to\infty} \int_a^b f(x) \cos nx dx = 0$. Hence evaluate $\lim_{n\to\infty} \int_0^1 x^2 \sin^2 nx \, dx$. [6]

Q.4

(c) Let f and g be analytic at $a \in \mathbb{C}$. Suppose that a is a zero of f order 2 and a is a zero of g order 4. Show that a is a pole of $\frac{f}{g}$ of order 2. Also, find the residue of $\frac{f}{g}$ at a. State the results you use. [6]

OR

(c) Let f be an entire function. Suppose that for given $a \in \mathbb{C}$ there is $n \in \mathbb{N} \cup \{0\}$ such that $f^{(n)}(a) = 0$. Show that f is a polynomial. Is the converse true? Justify. [6]
(d) Let f be an entire function, and let α, A and B be positive constants. If $|f(z)| \leq A|z|^\alpha + B$ for all $z \in \mathbb{C}$, then show that f is a polynomial of degree at most α. State the results you use. [6]

Q.5

(e) Let X be a non-empty set and $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on X generated by bases $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. Show that $\mathscr{T}_1 \supseteq \mathscr{T}_2$ if and only if given $B_2 \in \mathscr{B}_2$ and $x \in B_2$, there exists $B_1 \in \mathscr{B}_1$ such that $x \in B_1 \subseteq B_2$. Hence, compare the usual topology and the lower limit topology on $\mathbb{R}$. [6]

OR

(e) If $(X, \mathscr{T})$ is a Hausdorff topological space, then show that every sequence in X has at most one limit. Does the result hold if $(X, \mathscr{T})$ is not Hausdorff? Justify. [6]
(f) Define a topological property. Show that connectedness and compactness are topological properties. [6]

Q.6

(g) Show that a group of order 1225 is abelian. For primes p and q, is every group of order p^2q^2 abelian? Justify. [6]

OR

(g) Define a simple group. Show that a group of order pqr cannot be simple, where p, q, r are distinct primes such that $p < q < r$. [6]
(h) Let G be a group of order n and $x \in G$ such that $o(x) \neq 2$. Show that there exists a non-trivial automorphism of G. [6]

X