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No of printed pages: 2

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M.Sc. (Sem-III), PS03EMTH37, Mathematical Probability Theory;
Thursday, 1st November, 2018; 02.00 p.m. to 05.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. Let $(\Omega, \mathcal{A}, P)$ be probability space and $A, B \in \mathcal{A}$. Which one from following is true ?
   (A) $P(A \cap B) \leq P(A) + P(B) - 1$ (B) $P(A \cap B) \geq P(A) + P(B) - 1$
   (C) $P(A \cup B) = P(A) + P(B) - 1$ (D) none of these
2. Let A, B and C be independent events with $P(A) = 0.5, P(B) = 0.6$ and $P(C) = 0.1$. Then $P(A^C \cup B^C \cup C) =$
   (A) 0.27 (B) 0.73 (C) 0.71 (D) 0.69
3. Let F be a distribution function of rv X. Then for any $a, b \in \mathbb{R}$ with $a < b$, $P(a < X \leq b)$ is
   (A) $F(a) - F(b)$ (B) $F(b) - F(a)$ (C) $a - b$ (D) $b - a$
4. $\text{Var}(2 - X) =$
   (A) $\text{Var}(-X)$ (B) $\text{Var}(X)$ (C) $2 - \text{Var}(X)$ (D) $2 + \text{Var}(X)$
5. $X_n \xrightarrow{P} 0 \Leftrightarrow$
   (A) $\lim_{n\to\infty} E(\frac{|X_n|}{1+|X_n|}) = 0$ (B) $\lim_{n\to\infty} E(X_n) = 0$
   (C) $\lim_{n\to\infty} E(\frac{|X_n|}{1-|X_n|}) = 0$ (D) none of these
6. Let X_n be a sequence of independent random variables having values in $[0, \frac{1}{4}]$. Then $Z_n = X_1 \cdot X_2 \cdots X_n \xrightarrow{P}$
   (A) 0 (B) $\frac{1}{4}$ (C) 1 (D) none of these
7. If $\varphi(u)$ is characteristic function of random variable X, then the characteristic function of $2 - X$ is
   (A) $e^{2iu}\varphi(-u)$ (B) $e^{-2iu}\varphi(u)$ (C) $\varphi(-2u)$ (D) $\varphi(2 - u)$
8. Which inequality used for proving Strong Law of Large Numbers ?
   (A) Chebyshev's inequality (B) Jenson's inequality
   (C) Holder's inequality (D) Minkowski's inequality

Q.2 Attempt any ***seven***: [14]

(a) Let X be a discrete random variable having pmf, $p_X(x) = \begin{cases} \frac{k(x+1)}{5^x}, x \in \mathbb{N} \cup \{0\} \\ 0, \text{ otherwise} \end{cases}$. Find k.

(b) Let $f(x) = ae^{-b|x|}, x \in \mathbb{R}, a, b > 0$. Determine the relation between a and b so that f is a pdf of some rv.

(c) Let X and Y be rvs having joint pdf $f_{XY}(x, y) = \begin{cases} x + y, 0 < x, y < 1 \\ 0, \text{ otherwise} \end{cases}$.
Find $E(XY^2)$.

(d) State and prove Chebyshev's inequality.

(e) Prove: $|E(Z)| \leq E|Z|$, where Z is a complex rv.

(f) Define convergence almost surely.

(g) Let X_n be a sequence of random variables with $E(X_n) = 3$ and $\text{Var}(X_n) = \frac{1}{n}$, $\forall n$. Does X_n converge in probability ? Justify.

(h) State Lindeberg-Levy's theorem.

(i) Show that the characteristic function is continuous.

(P.T.O)

Q.3

(a) Let X be non negative extended random variable on $(\Omega, \mathcal{A})$. Then show that there exists an increasing sequence $\{X_n\}$ of non negative simple random variables on $(\Omega, \mathcal{A})$ such that $X_n(\omega) \to X(\omega), \forall\, \omega \in \Omega$. [6]

(b) Let $Z = (X, Y)$ be vector random variable. Show that $Z^{-1}(\mathcal{B}_2) = \sigma\left(X^{-1}(\mathcal{B}) \cup Y^{-1}(\mathcal{B})\right)$ where $\mathcal{B}_2$ and $\mathcal{B}$ are Borel σ-algebras in $\mathbb{R}^2$ and $\mathbb{R}$ respectively. [6]

OR

(b) Let X be a discrete rv having pmf $p_X(0) = \frac{1}{8}$, $p_X(1) = \frac{1}{4}$, $p_X(2) = \frac{1}{4}$, $p_X(-1) = \frac{1}{8}$, $p_X(-2) = \frac{1}{4}$, $p_X(x) = 0,\ x \neq -2, -1, 0, 1, 2$. Find pmf of a rv $Y = (X + 1)^2$.

Q.4

(a) State and prove C_r inequality. [6]

(b) Let D be dense subset of $\mathbb{R}$ and $F_D : D \to [0, 1]$ be non decreasing with [6]

$F_D(\infty) = 1, F_D(-\infty) = 0$. Define $F : \mathbb{R} \to [0, 1]$, $F(x) = \begin{cases} \inf\limits_{y>x} F_D(y), y \in D, x \in D^C \\ F_D(x), \quad x \in D \end{cases}$.

Show that F is a distribution function.

OR

(b) Let X & Y be non negative rvs. Then show that $E(aX + bY) = aE(X) + bE(Y)$, $a, b \in \mathbb{R}$. State results which you use.

Q.5

(a) Prove: $X_n \xrightarrow{P} X \Rightarrow f(X_n) \xrightarrow{P} f(X)$, where f is continuous function. [6]

(b) If $X_n \xrightarrow{P} X$ then show that $X_n \xrightarrow{L} X$. Does the converse hold ? Justify. [6]

OR

(b) State and prove Fatou's theorem.

Q.6

(a) State and prove Helly-Bray theorem. [6]

(b) Let $\varphi(u)$ be the characteristic function of rv X and F be the distribution function of X. If a, b $(a < b)$ are points of continuity of F then show that [6]

$$F(b) - F(a) = \lim_{U \to \infty} \frac{1}{2\pi} \int_{-U}^{U} \frac{e^{-iua} - e^{-iub}}{iu} \varphi(u)du.$$

OR

(b) Find probability density functions of random variables whose characteristic functions are (i) $e^{-\frac{1}{2}u^2}$ (ii) $e^{-|u|}$.

X

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