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Seat No.: _____ No. of printed pages: 2

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - III Examination
Saturday, 03rd November, 2018
PS03EMTH08, Group Theory

Time: 02:00 p.m. to 05:00 p.m. **Maximum marks: 70**

Note: Figures to the right indicate full marks of the respective questions. Assume standard notations wherever applicable.

Q-1 Write the question number and appropriate **option number only** for each question. [8]

(a) $\phi(\_\_\_) = \phi(\_\_\_)$, where ϕ denotes Euler's totient function.
(i) 2, 6 (ii) 5, 9 (iii) 7, 11 (iv) 8, 12

(b) $\mathbb{Z}/7\mathbb{Z}$ is isomorphic to _____.
(i) S_7 (ii) $\mathbb{Z}_7$. (iii) $\mathbb{Z}^7$ (iv) none of these

(c) Order of automorphism group $\mathscr{A}(\mathbb{Z}_2)$ of $\mathbb{Z}_2$ is _____.
(i) 1 (ii) 2 (iii) 3 (iv) 4

(d) If $e \neq a \in$ _____, then $c_a \neq 1$.
(i) S_2 (ii) S_9 (iii) $\mathbb{Z}_3$ (iv) $\mathbb{Z}$

(e) A group of order _____ is simple.
(i) 11 (ii) 21 (iii) 121 (iv) 221

(f) The order of 11-Sylow subgroups of S_{25} is _____.
(i) 11 (ii) 121 (iii) 1331 (iv) none of these

(g) A p-Sylow subgroup H of a group G is unique if H _____.
(i) is nonabelian (ii) is abelian (iii) $\subset Z(G)$ (iv) is finite

(h) Number of nonisomorphic abelian groups of order 128 is _____.
(i) 5 (ii) 15 (iii) 20 (iv) 64

Q-2 Attempt ***Any Seven*** of the following: [14]

(a) List invertible elements of the group $\mathbb{Z}_6$.

(b) Show that for two elements $f, g \in H = \{f \in A(\mathbb{R}) : f(0) = 1\}$, $f \circ g \notin H$.

(c) For a group G and $g \in G$, define $T_g : G \to G$ by $T_g(x) = gxg^{-1}$, $(x \in G)$. Show that $T_g \in \mathscr{A}(G)$.

(d) For a group G, define $\psi : G \to \mathscr{I}(G)$ by $\psi(g) = T_g$. Show that $\ker(\psi) = Z$.

(e) Mention the number of conjugates of (1 2)(2 3) in S_3 and S_4.

(f) Show that $S_7 \approx \{\tau \in S_9 : i\tau = i \text{ for all } i > 7\}$.

(g) Define a *solvable* group and give an example of the same.

(h) Show that S_3 is not an internal direct sum of its proper subgroups.

(i) If a group $G = N_1 \cdot N_2 = N_3 \cdot N_4$, then prove or disprove that $N_1 \approx N_3$ or $N_1 \approx N_4$.

1 (P.T.O.)

Q-3 (j) Let H, K be subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$. [6]

(k) Let N be a subgroup of a group G. Show that N is normal if and only if every right coset of N in G is also a left coset of N in G. [6]

OR

(k) Show that every cycle can be written as a product of transpositions. [6]

Q-4 (l) For a finite group G, show that "*being conjugate to*" is an equivalence relation on G. Also prove that $c_a = o(G)/o(N(a))$. [6]

(m) Let G be a group, $a \in G$, and $\phi \in \mathscr{A}(G)$. Show that $o(a) = o(\phi(a))$. [6]

OR

(m) Let p be a prime number and G be a group of order p^n for some $n \in \mathbb{N}$. Then show $Z(G) \neq \{e\}$. [6]

Q-5 (n) Let p be a prime and $n, m, r, \alpha \in \mathbb{N}$ such that $n = p^\alpha m, p^r \mid m$ but $p^{r+1} \nmid m$. Show that $p^r \mid \binom{p^\alpha m}{p^\alpha}$ but $p^{r+1} \nmid \binom{p^\alpha m}{p^\alpha}$. [6]

(o) Let G be a finite group and p be a prime such that $p^m | o(G)$ but $p^{m+1} \nmid o(G)$ for some integer $m \geq 1$. Using the class equation, prove that G has a subgroup of order p^m. [6]

OR

(o) If A, B are finite subgroups of a group G, then prove that $o(AxB) = \frac{o(A)o(B)}{o(A \cap xBx^{-1})}$. [6]

Q-6 (p) Suppose G is the internal direct product of $N_1, N_2, \ldots, N_n$. Show that $N_i \cap N_j = \{e\}$ for $i \neq j$. Also if $a \in N_i$, $b \in N_j$ then show that $ab = ba$. [6]

(q) If G and G' are isomorphic abelian groups, then show that for every integer s, $G(s)$ and $G'(s)$ are isomorphic. [6]

OR

(q) If two finite abelian groups are isomorphic, then show that they have the same invariants. [6]

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