[232] SEAT No.________

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SARDAR PATEL UNIVERSITY
M. Sc. (Semester III) Examination

Date: 26-10-2018, Friday **Time: 2.00 To 5.00 p.m.**

Subject: MATHEMATICS **Paper No. PS03CMTH23 – (Functional Analysis II)**

Total Marks: 70

Note: Throughout the paper, X and Y denote nlspaces.

1. Choose the correct option for each question: [8]

(1) Let Y be a subspace of X. Which of the following is true?
(a) If $Y \neq X$, then $Y^0 \neq \emptyset$ (c) If $Y^0 \neq \emptyset$, then $Y \neq X$
(b) If $Y \neq X$, then $Y^0 = \emptyset$ (d) If $Y = X$, then $Y^0 = \emptyset$

(2) Every linear functional on X is continuous, if X =
(a) l^1 (b) C[0, 1] (c) $\mathbb{C}^n$ (d) none of these

(3) Which of the following space is complete in $\| \ \|_\infty$?
(a) c_0 (b) c_{00} (c) $C^1[a, b]$ (d) none of these

(4) A map F: $(C^1[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$ defined by $F(x) = x'$ is
(a) closed and continuous (c) closed but not continuous
(b) continuous but not closed (d) none of these

(5) If $p < r < \infty$, then which of the following is true?
(a) $l^p \subset l^\infty$ (b) $l^r \subset l^p$ (c) $l^\infty \subset l^r$ (d) $l^p = l^r$

(6) For $x \in X$, let $j_x: X' \to K$ be defined by $j_x(f) = f(x)$. Then $\|j_x\| =$
(a) 1 (b) $\|x\|$ (c) $\|f\|$ (d) none of these

(7) Let F ∈ BL(X,Y). Then
(a) $\|F\| > \|F'\|$ (b) $\|F\| < \|F'\|$ (c) $\|F\| = \|F'\|$ (d) none of these

(8) Let I be the identity operator on X. Then $\sigma(I) =$ Let F ∈ BL(X,Y). Then
(a) $\emptyset$ (b) {0,1} (c) {0} (d) {1}

2. Attempt any SEVEN: [14]

(a) Prove: If $E \subset X$ is convex, then E^o is convex.
(b) Show that $\| \ \|_1$ and $\| \ \|_2$ are comparable norms on K^n.
(c) Show that $\mathbb{R}^2$ with $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$, is not strictly convex.
(d) Let X be a Banach space. If a series $\sum_n x_n$ of elements of X is absolutely summable, then show that it is summable in X.
(e) Prove: If a projection P on X is a closed map, then $R(P)$ and $Z(P)$ are closed in X.
(f) State Open mapping theorem.
(g) Let A ∈ BL(X). Show that $\sigma_e(A) \subset \sigma_a(A)$.
(h) Prove: If $F, G \in BL(X,Y)$, then $(F + G)' = F' + G'$.
(i) Let $F: X \to Y$ be a linear map. Prove that if F is continuous, then it sends every Cauchy sequence in X to Cauchy sequence in Y.

(PTO)

3. (a) Let Y be a closed subspace of a nls X. Show that, for $x+Y \in X/Y$, $|||x+Y||| = inf\{\|x+y\|: y \in Y\}$ defines a norm on X/Y. [6]

(b) Let $F: X \to Y$ be a linear map. Prove that F is continuous if and only if $Z(F)$ is closed in X and a linear map $\tilde{F}: X/Z(F) \to Y$ defined by $\tilde{F}(x+Z(F)) = F(x)$, $x \in X$ is continuous. [6]

OR

(b) Let $F: (\mathbb{R}^2, \|\ \ \|_1) \to \mathbb{R}$ be defined by $F(x(1), x(2)) = 3x(1) + 2x(2)$. Show that F is bounded and find $\|F\|$. [6]

4. (a) State and prove Hahn-Banach extension theorem. [6]

(b) Prove: If $BL(X,Y)$ is a Banach space, then Y is complete. [6]

OR

(b) Prove: The space $(C[0,1], \|\ \ \|_\infty)$ is complete. [6]

5. (a) Let X be a Banach space and Y be a nls. Let $F_n \in BL(X,Y), \forall n$. Suppose that for each $x \in X$, sequence $\{F_n(x)\}$ converges in Y. Define $F: X \to Y$ by $F(x) = \lim_n F_n(x)$. Show that, if E is totally bounded subset of X, then $\{F_n(x)\}$ converges uniformly to $F(x)$ on E. [6]

(b) Prove: If X and Y are Banach spaces and F: X →Y is a closed linear map, then F is continuous. [6]

OR

(b) Let $F: X \to Y$ be linear map. Prove that if there exists $\gamma > 0$ such that for every $y \in Y$, there is $x \in X$ with $F(x) = y$ and $\|x\| \le \gamma \|y\|$, then F is an open map. [6]

6. (a) Let $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $y \in Y$, let $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j)$, $x \in l^p$. Prove that $f_y \in (l^p)'$ and $\|f_y\| = \|y\|_q$. [6]

(b) Let $A \in BL(X)$ and $\dim X = \infty$. Show that $\sigma_e(A) = \sigma(A)$. [6]

OR

(b) Let $X = (l^p, \|\ \ \|_p), 1 \le p < \infty$. Define $A: X \to X$ by $A(x(1), x(2), \dots, x(n), \dots) = (0, x(1), \dots, x(n-1), \dots)$. Show that $\sigma_e(A) = \emptyset$. [6]

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