[202]

SEAT No.________

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Sardar Patel University

Mathematics

M.Sc. Semester III

Monday, 22 October 2018

2.00 p.m. to 5.00 p.m.

PS03CMTH21 - Real Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Which of the following is true?

(a) If $(X, \mathscr{A}, \mu)$ is saturated, then it is σ- finite.
(b) If $(X, \mathscr{A}, \mu)$ is saturated, then it is finite.
(c) If $(X, \mathscr{A}, \mu)$ is finite, then it is saturated.
(d) $(\mathbb{R}, \mathfrak{M}, m)$ is not saturated.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = e^{-x^2}$. Then the value of $\int_{\mathbb{R}} f d\delta_0$ is

(a) 0 (b) 1 (c) e (d) $\frac{1}{e}$

(3) Let m be the Lebesgue measure. Consider the signed measure $\nu = \delta_0 - m$ on $(\mathbb{R}, \mathfrak{M})$. Then the value of $\sup\{\nu(E) : E \text{ is a positive set}\}$ is

(a) 1 (b) -1 (c) 0 (d) ∞

(4) Which of the following is a pair of mutually singular measures on $(\mathbb{R}, \mathfrak{M})$?

(a) m, η (b) η, δ_0 (c) m, δ_0 (d) none of these

(5) Let $(X, \mathscr{A}, \mu)$ be a finite measure space and $1 < p < r < \infty$. Which of the following is true?

(a) $L^p(\mu) \subset L^r(\mu)$ (b) $L^p(\mu) \supset L^r(\mu)$ (c) $L^p(\mu) = L^r(\mu)$ (d) none of these

(6) If f is a continuous function on $\mathbb{R}$, then which of the following true with respect to the Lebesgue measure m?

(a) f is essentially bounded
(b) f is square integrable
(c) f is integrable
(d) none of these

(7) Let η be the counting measure and δ_0 be the Dirac measure concentrated at 0. Which of the following is an outer measure on $\mathbb{R}$?

(a) m (b) $m + \delta_0$ (c) $\delta_0 - \eta$ (d) $\eta + \delta_0$

(8) Let μ^* be an outer measure on X and $E \subset F \subset X$. Which of the following is true?

(a) $\mu^*(E) \leq \mu^*(F)$
(b) $\mu^*(F - E) = \mu^*(F) - \mu^*(E)$
(c) $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$
(d) $\mu^*(F) = \mu^*(F - E) + \mu^*(E)$

(P.T.O)

Q.2 Attempt any ***Seven.*** [14]

(a) Let $(X, \mathscr{A}, \mu)$ be a measure space. If $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ and $t = \sum_{j=1}^{m} \beta_j \chi_{B_j}$ are nonnegative measurable functions on X and if $s \leq t$, then show that $\int_X s d\mu \leq \int_X t d\mu$.

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space. If f is integrable over X, then show that f is finite a.e. $[\mu]$ on X.

(c) Define a complete measure space and give an example of a measure space which is not complete.

(d) If $\{A, B\}$ and $\{A_1, B_1\}$ are Hahn decomposition of $(X, \mathscr{A}, \nu)$, then show that $A \Delta A_1$ is a null set.

(e) Let ν be a signed measure and μ be a measure on $(X, \mathscr{A})$. If $\nu \perp \mu$ and $\nu \ll \mu$, then show that $\nu = 0$.

(f) Let $1 \leq p < \infty$. If $f, g \in L^p(\mu)$, then show that $f + g \in L^p(\mu)$ (do not use Minkowski's inequality).

(g) If f and g are measurable, $f = g$ a.e. $[\mu]$ on X and if f is essentially bounded, then show that g is essentially bounded.

(h) Let μ^* be an outer measure. If E_1 and E_2 are μ^*- measurable subsets of X and if $E_1 \cap E_2 = \emptyset$, then show that $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$.

(i) Let μ be a measure on an algebra $\mathscr{A}$ of subsets of X. If $\{A_i\} \subset \mathscr{A}$, $A \in \mathscr{A}$ and $A \subset \bigcup_i A_i$, then show that $\mu(A) \leq \sum_i \mu(A_i)$.

Q.3

(a) Let $(X, \mathscr{A}, \mu)$ be a measure space. If f is an integrable function on X and if $\{E_n\}$ is a sequence of pairwise disjoint measurable subsets of X, then prove in detail that [6]

$$\int_{\bigcup_{n=1}^{\infty} E_n} f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu.$$

(b) Let $(X, \mathscr{A})$ be a measurable space and f be a nonnegative measurable function on X. Show that there is sequence $\{s_n\}$ of nonnegative measurable simple functions such that $s_n \leq s_{n+1}$ for all n and $s_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$. [6]

OR

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space. Let a sequence $\{f_n\}$ of measurable functions such that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$. If there exists an integrable function g such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then show that $\int_X f_n d\mu \to \int_X f d\mu$ as $n \to \infty$. [6]

Q.4

(c) Let $(X, \mathscr{A}, \mu)$ be a σ- finite measure space, and let f and g be nonnegative measurable functions on X. If $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathscr{A}$, show that $f = g$ a.e. $[\mu]$ on X. [6]

(d) Let ν_1 and ν_2 be finite signed measures on $(X, \mathscr{A})$ and $\alpha \in \mathbb{R}$. Show that $|\alpha\nu_1| = |\alpha||\nu_1|$ and $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. [6]

OR

(d) If ν and μ are σ- finite measures on a measurable space $(X, \mathscr{A})$, then show that there exists a unique pair of measures ν_0 and ν_1 such that $\nu_0 \perp \mu$, $\nu_1 \ll \mu$ and $\nu = \nu_0 + \nu_1$. [6]

Q.5

(e) If $1 \leq p < \infty$, then show that $L^p(\mu)$ is complete. [6]

(f) Let $(X, \mathscr{A}, \mu)$ be a measure space. When is a measurable function called essentially [6] bounded? If f and g are essentially bounded and if α, β in $\mathbb{R}$, then show that both $\alpha f + \beta g$ and fg are essentially bounded.

OR

(f) Let $1 \leq p < \infty$ and $q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $(X, \mathscr{A}, \mu)$ be a finite measure space. If F is a continuous linear functional on $L^p(\mu)$, then prove that there is unique $g \in L^q(\mu)$ such that [6]

$$F(f) = \int_X fg d\mu \qquad (f \in L^p(\mu)).$$

Q.6

(g) Let μ be a σ- finite measure on an algebra $\mathcal{A}$, and let μ^* be the induced outer measure. Let $\mathbb{B}$ be the σ- algebra of all (μ^*)- measurable subsets of X, and let $\mathbb{B}'$ be the smallest σ- algebra of subsets of X containing $\mathcal{A}$. Show that the restriction of $\overline{\mu}$ to $\mathbb{B}'$ is the unique extension of μ to $\mathbb{B}'$. [6]

(h) Let μ^* be an outer measure on X. Show that the collection of all μ^*- measurable subsets of X is a σ- algebra. [6]

OR

(h) Let F be a cumulative distribution function of a finite Baire measure μ on a Borel σ- algebra on $\mathbb{R}$. Then prove that following statements. [6]

(A) F is bounded and increasing.

(B) F is right continuous on $\mathbb{R}$.

(C) F is left continuous at $x \in \mathbb{R}$ if and only if $\mu(\{x\}) = 0$.

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