[201/A53]

SEAT No.________

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Sardar Patel University
Mathematics
M.Sc. Semester III
Monday, 22 October 2018
2.00 p.m. to 5.00 p.m.
PS03CMTH01 - Real Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let η be the counting measure on $(\mathbb{R}, \mathfrak{M})$. Then the value of $\eta(\mathbb{Q})$ is

(a) 0 (b) 1 (c) 2 (d) ∞

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. Then the value of $\int_{\mathbb{R}} f d\delta_0$ is

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

(3) Which of the following is a signed measure but not a measure on $(\mathbb{R}, \mathfrak{M})$?

(a) m (b) $\eta - \delta_0$ (c) $\eta + \delta_0$ (d) $m - \delta_0$

(4) Let A and B be positive set of the signed measure ν on $(X, \mathscr{A})$? Which of the following may not be a positive set?

(a) $A \cup B$ (b) $A \cap B$ (c) $A \Delta B$ (d) $X - (A \cup B)$

(5) Which of the following $f : \mathbb{R} \to \mathbb{R}$ is essentially bounded with respect to the Lebesgue measure m?

(a) f is continuous
(b) $f \in L^2(\mu)$
(c) f is differentiable
(d) none of these

(6) Let $F : L^1(\mathbb{R}) \to \mathbb{R}$ be defined as $F(x) = \int_{\mathbb{R}} f(x) dm(x)$. Then the value of $\|F\|$ is

(a) 0 (b) 1 (c) ∞ (d) 2

(7) Let μ^* be an outer measure on X, and let $\{E_n\}$ be a sequence of pairwise disjoint subsets of X. Which of the following is true?

(a) $\mu^*(\bigcup_n E_n) \leq \sum_n \mu^*(E_n)$
(b) $\mu^*(\bigcup_n E_n) \geq \sum_n \mu^*(E_n)$
(c) $\mu^*(\bigcup_n E_n) = \sum_n \mu^*(E_n)$
(d) none of these

(8) Let F be the cumulative distribution function of a finite Baire measure μ on $(\mathbb{R}, \mathscr{B})$. Which of the following is true?

(a) F is unbounded
(b) F is differentiable
(c) $\lim_{x \to \infty} F(x) = 0$
(d) $\lim_{x \to -\infty} F(x) = 0$

Q.2 Attempt any *Seven*. [14]

(a) Let f be a function such that both $|f|$ and f^2 are measurable. Show that f need not be measurable.

(b) Show that every σ- finite measure space is saturated.

(c) Show that $(X, P(X), \eta)$ is a complete measure space.

(d) Let $(X, \mathscr{A}, \nu)$ be a signed measure space. If $A \in \mathscr{A}$ and $\nu(A) < 0$, then show that A may not be a negative set.

(e) If ν_1 and ν_2 are finite signed measures on $(X, \mathscr{A})$, then show that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

(f) If f is essentially bounded, then show that f is finite a.e. $[\mu]$ on X.

(P.T.O.)

(g) If f and g are measurable, $f = g$ a.e. $[\mu]$ on X and if $f \in L^p(\mu)$, then show that $g \in L^p(\mu)$, where $1 < p < \infty$.

(h) Let μ^* be an outer measure. If E_1 and E_2 are μ^*- measurable subsets of X and if $E_1 \cap E_2 = \emptyset$, then show that $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$.

(i) Let μ^* be an outer measure on X. If $A \subset X$ and $\mu^*(A) = 0$, then show that A is μ^*-measurable.

Q.3

(a) Let $(X, \mathscr{A})$ be a measurable space, and let D be a dense subset of $\mathbb{R}$. Suppose that for each $\alpha \in D$ there is an associated $B_\alpha \in \mathscr{A}$ such that $B_\alpha \subset B_{\alpha'}$ whenever $\alpha < \alpha'$. Show that there is a unique measurable function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on B_α^c for every $\alpha \in D$. [6]

(b) Let $(X, \mathscr{A})$ be a measurable space and f be a nonnegative measurable function. Show that there is sequence $\{s_n\}$ of nonnegative measurable simple functions such that $s_n \leq s_{n+1}$ for all n and $s_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$. [6]

OR

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on X. Show that $\int_X (\lim_{n\to\infty} f_n) d\mu = \lim_{n\to\infty} \int_X f_n d\mu$. [6]

Q.4

(c) Let ν be a signed measure on a measurable space $(X, \mathscr{A})$, and let $E \in \mathscr{A}$ with $0 < \nu(E) < \infty$. Show that E contains a positive set A with $\nu(A) > 0$. [6]

(d) Let f be an integrable function on a measure space $(X, \mathscr{A}, \mu)$. Define ν on $\mathscr{A}$ by $\nu(E) = \int_E f d\mu$, $E \in \mathscr{A}$. Find a Hahn decomposition and the Jordan decomposition of ν. [6]

OR

(d) Let ν and μ be σ- finite measures on a measurable space $(X, \mathscr{A})$, and let $\nu \ll \mu$. If f is a nonnegative measurable function on X, then show that $\int_E f d\nu = \int_E f[\frac{d\nu}{d\mu}] d\mu$ for every $E \in \mathscr{A}$. [6]

Q.5

(e) Let $(X, \mathscr{A}, \mu)$ be a finite measure space, $1 \leq p < \infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that g is an integrable function on X satisfying $\left|\int_X g\varphi d\mu\right| \leq M\|\varphi\|_p$ for some $M > 0$ and for all measurable simple functions φ. Show that $g \in L^q(\mu)$. [6]

(f) Show that $(L^\infty(\mu), \|\cdot\|_\infty)$ is complete. [6]

OR

(f) Let $1 \leq p < \infty$. Let $f \in L^p(\mu)$, and let $\epsilon > 0$. Prove that there is a measurable simple function φ vanishing outside a set of finite measure such that $\|f - \varphi\|_p < \epsilon$. [6]

Q.6

(g) Let μ^* be an outer measure on X. Show that the collection $\mathbb{B}$ of all μ^*- measurable subsets of X is a σ- algebra. [6]

(h) Let μ be a σ- finite measure on an algebra $\mathcal{A}$ of subsets of X, and let μ^* be the outer measure induced by μ. Show that a subset E of X is (μ^*)- measurable if and only if E can be expressed as a difference $E = A - B$, where A is an $\mathcal{A}_{\sigma\delta}$- set and $\mu^*(B) = 0$. [6]

OR

(h) Let μ be a σ- finite measure on an algebra $\mathcal{A}$, and let μ^* be the induced outer measure. Let $\mathbb{B}$ be the σ- algebra of all (μ^*)- measurable subsets of X, and let $\mathbb{B}'$ be the smallest σ- algebra of subsets of X containing $\mathcal{A}$. Show that the restriction of $\overline{\mu}$ to $\mathbb{B}'$ is the unique extension of μ to $\mathbb{B}'$. [6] [6]

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