Seat No._____

[91 & A-11]

No of printed pages: 2

Sardar Patel University

M.Sc. Semester III Examination

Monday, 24^{th} October 2016

14.00 to 17.00

Subject: Mathematics

Paper Code: PS03EMTH02 **Title of the Paper: Banach Algebras**

Maximum Marks: 70

Note: $\mathcal{A}, \mathcal{B}$ denote complex Banach algebras with identity unless mentioned explicitly, other notations are usual.

Q.1 Write the question number and correct **option number only** for each question. [8]

(*a*) The algebra _____ with convolution multiplication does not have identity.

(i) $\ell^1(\mathbb{Z})$ (ii) $L^1(\mathbb{R})$ (iii) $\ell^1(\mathbb{Z}_4)$ (iv) $\ell^1(\mathbb{Z}_7)$

(*b*) _____ $\subset$ _____.

(i) $C[0,1]$, $C[0,2]$ (ii) $C[0,2]$, $C[0,1]$ (iii) $B[0,1]$, $\mathcal{L}t([0,1])$ (iv) $C[0,1]$,$\mathcal{L}t([0,1])$

(*c*) For a homomorphism $f : \mathcal{A} \to \mathcal{B}$ and $x \in \mathcal{A}$, _____.

(i) $\mathrm{sp}_{\mathcal{A}}(x) \subset \mathrm{sp}_{\mathcal{B}}(x)$ (iii) $\mathrm{sp}_{\mathcal{A}}(x) \cap \mathrm{sp}_{\mathcal{B}}(x) = \emptyset$

(ii) $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_{\mathcal{B}}(x)$ (iv) $\mathrm{sp}_{\mathcal{A}}(x) \cap \mathrm{sp}_{\mathcal{B}}(x) \neq \emptyset$

(*d*) $\mathbb{C}^3$ with pointwise operations has _____ maximal ideals.

(i) 3 (ii) 6 (iii) 9 (iv) 12

(*e*) $\mathrm{Rad}(\mathbb{C}^n)$ with pointwise operations is _____.

(i) $\{0\}$ (ii) $\mathbb{C}^n$ (iii) $\mathbb{C}^{n-1}$ (iv) $\mathbb{C}$

(*f*) Which of the following is a closed ideal of $C^7[0,1]$? $\{f \in C^7[0,1] :$ _____$\}$.

(i) $f' = 0$ (ii) $f'' = 0$ (iii) $f'(0) = f'(1) = 0$ (iv) $f(0) = f'(0) = 0$

(*g*) Gel'fand transform of $\mathcal{A}$ is an isometry if _____.

(i) $\|x^2\| = \|x\|^2$ for all $x \in \mathcal{A}$ (iii) $\mathcal{A} = \ell^1$

(ii) $\mathcal{A} = M_2(\mathbb{C})$ (iv) $\dim(\mathcal{A}) < \infty$

(*h*) _____ is a C^*-algebra.

(i) $\mathcal{P}[0,1]$ (ii) $C(\mathbb{R})$ (iii) $\mathbb{C}$ (iv) ℓ^p

Q.2 Attempt any Seven. **(Start a new page.)** [14]

(*a*) Show that multiplication on $\mathcal{A}$ is jointly continuous.

(*b*) Find the norm of $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ in the C^*-algebra $M_2(\mathbb{C})$.

(*c*) Find the spectral radius of $f \in C^1[0,1]$, where $f(t) = \cos(\sin(-\pi t))$, $(t \in [0,1])$.

(*d*) Define a *complex homomorphism* on a Banach algebra and give one example of the same.

(*e*) Give an example of an element x of a Banach algebra $\mathcal{A}$ such that $r(x) < \|x\|$.

(*f*) Define a *unitary* element of a C^*-algebra and give its example.

(*g*) If $x \in \mathrm{Rad}(\mathcal{A})$, then show that $r(x) = 0$.

(*h*) In a C^*-algebra $\mathcal{A}$, show that span of all normal elements is $\mathcal{A}$.

(*i*) Show that $\{f \in C[0,1] : f((0.5 - \varepsilon, 0.5)) = \{0\}$ for some $\varepsilon > 0\}$ is an ideal of $C[0,1]$.

[Contd...]

Q.3 (Start a new page.)

(a) Prove that $\mathcal{L}t(X) \subsetneq B(X)$.

(b) Define, and prove submultiplicativity of, a norm on $C^2[0,1]$ making it a Banach algebra. [6]

OR

(b) Define *topological divisor of zero* and prove its existence in every infinite dimensional Banach algebra with identity. [6]

Q.4 (Start a new page.)

(c) For $x \in \mathcal{A}$, prove that $\mathrm{sp}(x)$ is compact. Does the result hold if $\mathcal{A}$ is a real algebra? [6]

(d) Giving all details, prove that $\mathrm{Rad}(A)$ is a two sided ideal of $\mathcal{A}$. [6]

OR

(d) Define the Gel'fand transform of an element x of a commutative unital Banach algebra $\mathcal{A}$. Show that $\widehat{A}$ contains constants and separates the points of $m(A)$. [6]

Q.5 (Start a new page.)

(e) For a compact T_2 topological space X, let $\mathcal{I}$ be a closed ideal of $C(X)$. Show that there exists $t_0 \in X$ such that $f(t_0) = 0$ for all $f \in \mathcal{I}$. [6]

(f) Show that the maximal ideal space of $C[0,1]$ is homeomorphic to $[0,1]$ [6]

OR

(f) For compact T_2-spaces X, Y and a continuous function $h : X \to Y$, define $\psi : C(Y) \to C(X)$ by $\psi(f) = f \circ h$; $(f \in C(Y))$. Show that ψ is a continuous *-homomorphism. [6]

Q.6 (Start a new page.)

(g) Define an C^*-*algebra*. Prove that spectrum of a selfadjoint element of a C^*-algebra is real. [6]

(h) For a normal element x of a C^*-algebra $\mathcal{A}$, show that $\|x^2\| = \|x\|^2$. [6]

OR

(h) Define an involution on $A(\mathbb{D})$ making it a Banach*-algebra. Show that $A(\mathbb{D})$ is not a C^*-algebra. [6]

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