(101 & A-21) **Seat No.:**________ No of printed pages: 2

Sardar Patel University
Mathematics
M.Sc. Semester III
Friday, 21 October 2016
2.00 p.m. to 5.00 p.m.
PS03EMTH01 - Functional Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $1 \le p < q \le \infty$. Which of the following is true?

(a) $L^p[0,1] \subset L^q[0,1]$ (b) $L^p[0,1] \supset L^q[0,1]$ (c) $L^p[0,1] = L^q[0,1]$ (d) none of these

(2) Let E_1 and E_2 be subsets of a normed space X. Which of the following is true?
(a) If E_1 is open, then $E_1 \cup E_2$ is open.
(b) If E_2 is open, then $E_1 + E_2$ is open.
(c) $E_1 + E_2$ is closed if and only both E_1 and E_2 are closed.
(d) If E_1 is compact, then $E_1 \cup E_2$ is compact.

(3) Let $\{0\} \neq X$ and Y be normed spaces, and let $T : X \to Y$ be linear and continuous. Let $\alpha = \sup\{\|Tx\| : \|x\| < 1\}$ and $\beta = \sup\{\|Tx\| : \|x\| = 1\}$. Which of the following is true?

(a) $\alpha < \|T\|$ (b) $\alpha < \beta$ (c) $\alpha > \beta$ (d) none of these

(4) Let f be a nonzero linear functional on a normed space X. Which of the following is true?

(a) f is continuous
(b) f is onto
(c) f is a closed map
(d) none of these

(5) Which of the following is a Banach space with the sup norm?

(a) c_{00}
(b) $C^2[0,1]$
(c) $\mathbb{P}[0,1]$
(d) $\{f \in C[0,1] : f(\frac{3}{4}) = 0\}$

(6) Let $D_m(x) = \sum_{k=-m}^{m} e^{ikx}$. Then the value of $\lim_{n\to\infty} \widehat{D_m}(n)$ is

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) ∞

(7) Let T be the Fredholm integral operator with continuous kernel k? Which of the following is true?

(a) $\|T\| \le \|k\|_\infty$ (b) $\|T\| > \|k\|_\infty$ (c) $\|k \diamond k\|_\infty > \|k\|_\infty^2$ (d) none of these

(8) The dual $(c_{00}, \|\cdot\|_\infty)$ is isometrically isomorphic to

(a) ℓ^1 (b) c_0 (c) ℓ^∞ (d) ℓ^2

Q.2 Attempt any *Seven*. [14]

(a) Let Y be a closed subspace of a normed space X. If a sequence $(x_n + Y)$ in X/Y converges to $x + Y \in X/Y$, then show that there is a sequence (y_n) in Y such that $x_n + y_n \to x$.
(b) Show that the closure of a convex set in a normed space is a convex set.
(c) Let $(X, \|\cdot\|)$ be a normed space, and let f be a nonzero linear functional on X. Show that $f(E)$ is an open set for every open subset E of X.
(d) Let $f : (c_{00}, \|\cdot\|_\infty) \to \mathbb{K}$ be $f((x(k))) = \sum_{k=1}^{n} x(k)$. Find the norm of f.
(e) Let X and Y be normed space, and let $\mathscr{F} \subset BL(X,Y)$. If $\mathscr{F}$ is unbounded at some $x \in X$, then show that $\mathscr{F}$ is unbounded at every x in a dense subset of X.

(P.T.O.)

(f) Let Z be a closed subspace of a normed space X. Let $Q : X \to X/Z$ be $Q(x) = x + Z$. Show that Q is an open map.

(g) Let P be a projection on a normed space X. If P is closed, then show that both $Z(P)$ and $R(P)$ are closed in X.

(h) Define weak convergence of a sequence of a normed space. Show that weak limit of a sequence is unique.

(i) Let F' be the transpose of $F \in BL(X, Y)$. Show that $\|F'\| = \|F\|$.

Q.3

(a) Let $\{y_1, y_2, \ldots, y_m\}$ be a basis of a normed space X. For $n \in \mathbb{N}$, let $x_n = k_{n1}y_1 + k_{n2}y_2 + \cdots + k_{nm}y_m$ and $x = k_1y_1 + k_2y_2 + \cdots + k_my_m$, where k_{ij} and k_i are scalars. Show that $x_n \to x$ if and only if $k_{nj} \to k_j$ for all $j = 1, 2, \ldots, m$. Also, show that (x_n) is bounded if and only if each (k_{nj}) is bounded. [6]

(b) Let X be a normed space. If the set $\{x \in X : \|x\| \leq 1\}$ is compact, then show that X is finite dimensional. State the results you use. [6]

OR

(b) Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on a linear space X. When is $\|\cdot\|$ called stronger than $\|\cdot\|'$? Show that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if there are positive constants α and β such that $\alpha\|\cdot\| \leq \|\cdot\|' \leq \beta\|\cdot\|$. [6]

Q.4

(c) State and prove Hahn-Banach Separation Theorem. [6]

(d) Consider $X = \{x \in C[-\pi, \pi] : x(-\pi) = x(\pi)\}$ with the sup norm. Show that the Fourier series of every x in a dense subset of X diverges at 0. [6]

OR

(d) Let Y be a closed subspace of a normed space X. Show that X is a Banach space if and only if both Y and X/Y are Banach spaces. State the result you use. [6]

Q.5

(e) (क) Let X and Y be normed spaces, and let $F : X \to Y$ be linear. Suppose that $Z(F)$ is closed in X. Let $\widetilde{F} : X/Z \to Y$ be $\widetilde{F}(x + Z(F)) = F(x)$. Show that F is an open map if and only if $\widetilde{F}$ is an open map. [3]

(ख) If X is a Banach space with the norms $\|\cdot\|$ and $\|\cdot\|'$, then show that either $\|\cdot\|$ and $\|\cdot\|'$ are equivalent or they are not comparable. [3]

(f) State and prove Closed Graph Theorem. [6]

OR

(f) Show that $(C[a, b], \|\cdot\|_1)$ and $(C^1[a, b], \|\cdot\|_\infty)$ are not Banach spaces. [6]

Q.6

(g) Let X be a normed space. If X' is separable, then show that X is separable. Is the converse true? Justify. [6]

(h) Show that weak convergence implies weak*- convergence. Also show that every bounded sequence in X' has a weak*- convergent subsequence. [6]

OR

(h) (ग) Let $T \in BL(X, Y)$. Show that T is not bounded below if and only if there is a sequence (x_n) in X such that $\|x_n\| = 1$ for all n and $\|Tx_n\| \to 0$ as $n \to \infty$. [3]

(घ) Let $T \in BL(X)$. If $\lambda \in \sigma_a(T)$, then show that $|\lambda| \leq \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}$. [3]

ььььььььь

— X —