(57 & A-21) Seat No: ________

No of printed pages: 2

Sardar Patel University

Mathematics

M.Sc. Semester III

Monday, 17 October 2016

2.00 p.m. to 5.00 p.m.

PS03CMTH01 - Real Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $X = \{1,2,3\}$, and let $\mathcal{U} = \{\{1\},\{2\}\}$. Then the number of elements in the smallest σ-algebra on X containing $\mathcal{U}$ is

(a) 2 (b) 4 (c) 6 (d) 8

(2) Let $(X, \mathscr{A})$ be a measurable space, and let $f : X \to [-\infty, \infty]$ be a function. Which of the following implies that f is measurable?

(a) f^2 is measurable
(b) $|f|$ is measurable
(c) f is one one
(d) None of these

(3) Let δ_0 be the point mass measure at 0, and let m be the Lebesgue measure. Let $\nu = \delta_0 - m$. Then $\nu([-1,1]) = \dots\dots\dots$

(a) 1 (b) -1 (c) 2 (d) none of these

(4) Let ν be a signed measure and μ be a measure on $(X, \mathscr{A})$. Which of the following implies that $\nu = 0$?

(a) $\nu \ll \mu$ (b) $\nu \perp \mu$ (c) $\mu = 0$ (d) $\nu \ll \mu$ and $\nu \perp \mu$

(5) Let $f, g \in L^2(\mu)$. Then fg is in

(a) $L^1(\mu)$ (b) $L^2(\mu)$ (c) $L^\infty(\mu)$ (d) none of these

(6) The concept of product measure makes use of ....

(a) Radon-Nikodym Theorem
(b) Caratheodory's Extension Theorem
(c) Cumulative Distribution Function
(d) None of these

(7) If μ^* is an outer measure on X and $E \subsetneq F \subset X$, then which of the following is true?

(a) $\mu^*(E) < \mu^*(F)$ (b) $\mu^*(E) \leq \mu^*(F)$ (c) $\mu^*(E) \geq \mu^*(F)$ (d) $\mu^*(E) > \mu^*(F)$

(8) Suppose $f \in L^\infty(\mu)$ with $\|f\|_\infty = 1$. Let $g = f$ a.e.. Then

(a) $\|g\|_\infty \leq 1$ (b) $\|g\|_\infty \geq 1$ (c) $\|g\|_\infty = 1$ (d) none of these

Q.2 Attempt any *Seven.* [14]

(a) Show that a measurable set contained in a set of σ-finite measure is of σ-finite measure.

(b) Show that the composition of a continuous function with a measurable function is measurable.

(P.T.O.)

(c) Let f be a nonnegative measurable function on a measure space $(X, \mathscr{A}, \mu)$. If $\int_X f d\mu = 0$, then show that $f = 0$ a.e. $[\mu]$ on X.

(d) Let ν be a signed measure on a measurable space $(X, \mathscr{A})$. Show that $|\nu(E)| \leq |\nu|(E)$ for all measurable set E.

(e) Let ν, λ and μ be σ- finite measures on a measurable space $(X, \mathscr{A})$. If $\nu \ll \mu \ll \lambda$, then show that $\left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right]\left[\frac{d\mu}{d\lambda}\right]$.

(f) If f is an essentially bounded function on a measure space $(X, \mathscr{A}, \mu)$, then show that $|f(t)| \leq \|f\|_\infty$ a.e. $[\mu]$ on X.

(g) State a Density Theorem in L^p- spaces, $1 < p < \infty$.

(h) Show that the outer measure induced by a measure on an algebra is regular.

(i) If F is a cumulative distribution of a Baire measure μ, show that $\lim_{x \to -\infty} F(x) = 0$.

Q.3

(a) State and Prove Lebesgue Dominated Convergence Theorem. [6]

(b) Let $(X, \mathscr{A})$ be a measurable space, and let D be a dense subset of $\mathbb{R}$. Suppose that for each $\alpha \in D$ there is an associated $B_\alpha \in \mathscr{A}$ such that $B_\alpha \subset B_{\alpha'}$ whenever $\alpha < \alpha'$. Prove that there is a unique measurable function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on B_α^c for every $\alpha \in D$. [6]

OR

(b) State and prove Fatou's Lemma. Just state whether this result can be obtained from Monotone Convergence Theorem. [6]

Q.4

(c) State and prove Hahn Decomposition Theorem. [6]

(d) State and prove Lebesgue Decomposition Theorem. [6]

OR

(d) If ν is a signed measure on a measurable space $(X, \mathscr{A})$, then show that there exists unique measures ν_1 and ν_2 on $(X, \mathscr{A})$ such that $\nu = \nu_1 - \nu_2$ and $\nu_1 \perp \nu_2$. [6]

Q.5

(e) Suppose that $(X, \mathscr{A}, \mu)$ is a σ- finite measure space. Let $1 < p < \infty$, and let $q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q(\mu)$, define $F_g : L^p(\mu) \to \mathbb{R}$ by $F_g(f) = \int_X fg d\mu$ for all $f \in L^p(\mu)$. Prove that F_g is a continuous linear functional on $L^p(\mu)$ and $\|F_g\| = \|g\|_q$. [6]

(f) State and prove Minkowski's inequality. Further, discuss when equality holds. [6]

OR

(f) Show that $(L^\infty(\mu), \|\cdot\|_\infty)$ is a normed algebra. [6]

Q.6

(g) State and prove Caratheodory's Extension Theorem. [6]

(h) Prove that the restriction of the outer measure to the collection of all measurable sets is a complete measure. [6]

OR

(h) Suppose that μ is a measure on an algebra $\mathcal{A}$ and μ^* is the induced outer measure. Prove that $\mu^* = \mu$ on $\mathcal{A}$. Also, show that every member of $\mathcal{A}$ is measurable. [6]

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