Seat No. ______

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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - III Examination

Monday, 1st April, 2019

PS03EMTH36, Group Theory

Time: 02:00 p.m. to 05:00 p.m. **Maximum marks: 70**

Note: (1) Figures to the right indicate marks of the respective question.
(2) Assume usual/standard notations wherever applicable.

Q-1 Choose the most appropriate option for each of the following questions. [08]

1. The group ______ is abelian.
   (a) S_2 (b) S_3 (c) S_5 (d) S_6
2. Let G be a finite group and $a \in G$. Then $a^{2o(G)} =$ ______.
   (a) a (b) a^2 (c) e (d) none of these
3. The number of odd permutations in S_5 is ______.
   (a) 5 (b) 30 (c) 60 (d) 120
4. ______ is a valid class equation of a group of order 5.
   (a) $5 = 1 + 4$ (c) $5 = 1 + 2 + 2$
   (b) $5 = 2 + 3$ (d) $5 = 1 + 1 + 1 + 1 + 1$
5. If G is a group and p is a prime such that $p \mid o(G)$, then the number of p-Sylow subgroups of G ______.
   (a) is always more than p (c) divides p
   (b) is always less than p (d) does not divide p
6. The permutation group S_4 cannot have a ______-Sylow subgroup.
   (a) 2 (b) 3 (c) 5 (d) 8
7. The number of non-isomorphic abelian groups of order 108 is ______.
   (a) 3 (b) 5 (c) 6 (d) 7
8. The invariants of Klein-4 group are ______.
   (a) $2, 2$ (b) $1, 1$ (c) $2, 1$ (d) $2, 4$

Q-2 Attempt *any seven* of the following. [14]

(a) For $n \in \mathbb{Z}$, show that $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$ is an abelian group under addition.

(b) State Lagrange's theorem.

(c) Let G be a group and $T : G \to G$ be defined by $T(x) = x^{-1}$. Show that $T \in A(G)$.

(d) Given an example of a permutation in S_4 which is conjugate to $(2, 1, 4)$.

(e) Give an example of a simple group.

(f) Define a solvable group.

(g) Define a p-Sylow subgroup of a group for a prime p.

(h) Define internal direct product of groups.

(i) Let A and B be any two groups and e be the identity of A. Show that the set $\bar{B} = \{(e, b) \in G \mid b \in B\}$ is a normal subgroup of $A \times B$.

(P.T.O)

Q-3 (a) Let G be a finite abelian group and p be a prime such that $p \mid o(G)$. Show that there is $a \in G$, such that $o(a) = p$. [06]

(b) Let H and K be two subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$. [06]

OR

(b) Let G and G' be two groups and $\varphi : G \to G'$ be an onto homomorphism. Show that $G/\ker\varphi$ is isomorphic to G'. [06]

Q-4 (a) Let G be a group of order p^n, where p is a prime and $n \in \mathbb{N}$. Show that $|Z(G)| \neq 1$ and deduce that a group of order p^2 is abelian. [06]

(b) Define inner automorphism of a group G. If $\mathscr{I}(G)$ denotes the group of all inner automorphisms of G, then prove that $\mathscr{I}(G) \approx G/Z(G)$ [06]

OR

(b) Prove that every group is isomorphic to a subgroup of $A(S)$ for some set S. [06]

Q-5 (a) State and prove Sylow's theorem. [06]

(b) Show that there is no non-abelian group of order $11^2 \times 13^2$. [06]

OR

(b) Show that a group of order 72 cannot be simple. [06]

Q-6 (a) Let p be a prime. Prove that two groups of order p^n are isomorphic if they have the same invariants. [06]

(b) For an abelian group G and an integer s, let $G(s) = \{x \in G \mid x^s = e\}$, where e is identity of the group G. Prove that if G and G' are isomorphic abelian groups, then $\forall\, s \in \mathbb{Z}$, $G(s)$ and $G'(s)$ are isomorphic. [06]

OR

(b) Prove that every finite abelian group is the direct product of cyclic groups. [06]

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