Seat No. ______

[126]

No. of printed pages: 2

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - III Examination

Monday, 1st April, 2019

PS03EMTH08, Group Theory

Time: 02:00 p.m. to 05:00 p.m. **Maximum marks: 70**

Note: (1) Figures to the right indicate marks of the respective question.
(2) Assume usual/standard notations wherever applicable.

Q-1 Choose the most appropriate option for each of the following questions. [08]

1. The group ______ is non-abelian.
   (a) S_2 (b) S_3 (c) $\mathbb{Z}_5$ (d) $\mathbb{Z}_6$
2. For a set S with 3 elements, the cardinality of $A(S)$ is ______.
   (a) 3 (b) 5 (c) 6 (d) 9
3. The cycle $(1,4,5,7) \in S_7$ does not commute with ______.
   (a) $(3,6,5)$ (b) $(5,7,1,4)$ (c) $(6,2,3)$ (d) $(3,6)$
4. A 5-cycle in S_7 is ______ permutation
   (a) an odd (b) an even (c) not a (d) a trivial
5. Subgroup of order 3 in the permutation group S_3 is ______.
   (a) non-abelian (b) not simple (c) normal (d) not unique
6. The order of a 3-Sylow subgroup in the permutation group S_9 is ______.
   (a) 4 (b) 9 (c) 27 (d) 81
7. The number of non-isomorphic abelian groups of order 36 is ______.
   (a) 4 (b) 5 (c) 6 (d) 7
8. The invariants of a non-cyclic abelian group of order 9 are ______.
   (a) 9 (b) 1, 1 (c) 2 (d) 3, 3

Q-2 Attempt *any seven* of the following. [14]

(a) Let H be a subgroup of a group G. Define index of H in G.

(b) State Euler's theorem.

(c) Let G be a group and let $T : G \to G$ be defined by $T(x) = x^{-1}$. Show that T is a homomorphism if and only if G is abelian.

(d) Let G be a group and $a \in G$. Show that $N(a)$ is a subgroup of G.

(e) Define a solvable group.

(f) Give an example of a simple group.

(g) State Sylow's theorem.

(h) Let G be an abelian group of order p^n, where p is a prime and $n \in \mathbb{N}$. Define invariants of G.

(i) Let A and B be any two groups and f be the identity of B. Show that the set $\bar{A} = \{(a, f) \in G \mid a \in A\}$ is a subgroup of $A \times B$.

CP.T.O.

Q-3 (a) Let H and K be two subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$. [06]

(b) Let H, K be two subgroups of a finite group G. Show that $o(HK) = \dfrac{o(H)o(K)}{o(H \cap K)}$. [06]

OR

(b) Let S be a non-empty set. Show that the set of all bijective functions from S to S is a group under composition. [06]

Q-4 (a) Prove that if G is a non-abelian group and p is a prime such that $p \mid o(G)$, then G has an element of order p. [06]

(b) State and prove Cayley's theorem. [06]

OR

(b) For a prime p, let G be a group of order p^4. Show that $o(Z(G)) \neq p^3$. [06]

Q-5 (a) Prove that a unique p-Sylow subgroup of a group G is normal. [06]

(b) Show that a group of order $7^2 \times 11^2$ is abelian. [06]

OR

(b) Show that a group of order 80 cannot be simple. [06]

Q-6 (a) Let G be a group and suppose that G is the internal direct product of $N_1, N_2 \ldots, N_k$. Let $T = N_1 \times N_2 \times \cdots \times N_k$. Then prove that G is isomorphic to T. [06]

(b) For an abelian group G and an integer s, let $G(s) = \{x \in G \mid x^s = e\}$, where e is identity of the group G. Prove that if G and G' are isomorphic abelian groups, then $\forall\, s \in \mathbb{Z}$, $G(s)$ and $G'(s)$ are isomorphic. [06]

OR

(b) Describe all the finite abelian groups of order [06]

(i) 2^5. (ii) 7^4. (iii) $2^3 \cdot 3^2$.

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