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SARDAR PATEL UNIVERSITY

M. Sc. (Semester III) Examination

Date: 25-3-2019, Monday **Time: 2.00 To 5.00 p.m.**

Subject: MATHEMATICS **Paper No. PS03CMTH23 – (Functional Analysis II)**

Total Marks: 70

Note: Throughout the paper, X and Y denote nlspaces.

1. Choose the correct option for each question: [8]

(1) If $E \subset X$ is convex, then which of the following sets may not be convex?
(a) E^0 (b) $\bar{E}$ (c) E^c (d) none of these

(2) Every linear functional on X is continuous, if X =
(a) l^1 (b) $\mathbb{C}^n$ (c) C[0, 1] (d) none of these

(3) Let X & Y be nlspaces. Which of the following need not be a Banach space?
(a) BL(X, K) (b) BL(X,Y) (c) BL(X′,Y) (d) none of these

(4) A map F: $(C^1[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$ defined by F(x) = x′ is
(a) closed but not continuous (b) closed and continuous
(c) continuous but not closed (d) none of these

(5) If $p < r < \infty$, then which of the following is true?
(a) $l^p \subset l^\infty$ (b) $l^r \subset l^p$ (c) $l^\infty \subset l^r$ (d) $l^p = l^r$

(6) For x ∈ X, let j_x: X′ → K be defined by $j_x(f) = f(x)$. Then $\|j_x\| =$
(a) 1 (b) $\|f\|$ (c) $\|x\|$ (d) none of these

(7) If $dim\, X = n$, then $dim\, BL(X, K)$ is
(a) n^2 (b) n (c) $n - 1$ (d) $n + 1$

(8) Let I be the identity operator on X. Then $\sigma(I) =$
(a) $\emptyset$ (b) {0,1} (c) {0} (d) {1}

2. Attempt any SEVEN: [14]

(a) State Holder's inequality.
(b) Let $E_1, E_2 \subset X$ and E_1 be open in X. Show that $E_1 + E_2$ is open in X.
(c) Show that $\|.\|_1$ and $\|.\|_\infty$ are comparable norms on K^n.
(d) State Uniform Boundedness Principle.
(e) Let X be a Banach space. If a series $\sum_n x_n$ of elements of X is absolutely summable, then show that it is summable in X.
(f) Prove: If a projection P on X is a closed map, then $R(P)$ and $Z(P)$ are closed in X.
(g) For $A \in BL(X)$, define $\sigma_e(A)$, $\sigma_a(A)$ and show that $\sigma_e(A) \subset \sigma_a(A)$.
(h) Let $F \in BL(X, Y)$. Show that $\|F\| = \|F'\|$.
(i) Let $F: X \to Y$ be a linear map. Prove that if F is continuous, then it sends every Cauchy sequence in X to Cauchy sequence in Y.

(P.T.O.)

3. (a) Let Y be a closed subspace of a nls X. Show that, for $x + Y \in X/Y$, $|||x + Y||| = inf\{||x + y||: y \in Y\}$ defines a norm on X/Y. [6]

(b) Let $F \in BL(X,Y)$. Define a map $\widetilde{F}: X/Z(F) \rightarrow Y$ by $\widetilde{F}(x + Z(F)) = F(x)$. Show that $\widetilde{F}$ is linear and continuous. [6]

OR

(b) Prove: If $dimX < \infty$, then every linear map from X to Y is continuous. [6]

4. (a) Prove: A Banach space cannot have a denumerable basis. [6]

(b) State the Hahn-Banach extension theorem and show that the Hahn-Banach extension may not be unique. [6]

OR

(b) Prove: The space $(C[0,1], \|.\|_{\infty})$ is complete. [6]

5. (a) Prove: A subset $E \subset X$ is bounded if and only if $f(E)$ is bounded in K, for every $f \in X'$. [6]

(b) Prove: If X and Y are Banach spaces and $F: X \rightarrow Y$ is a closed linear map, then F is continuous. [6]

OR

(b) State and prove Open mapping theorem. [6]

6. (a) Let $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $y \in Y$, let $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j)$, $x \in l^p$. Prove that $f_y \in (l^p)'$ and $\|f_y\| = \|y\|_q$. [6]

(b) Let $F, G \in BL(X, Y)$ & $k \in K$. Prove that $(F + G)' = F' + G'$ and $(kF)' = kF'$. [6]

OR

(b) Let X be a nls and $A \in BL(X)$. Prove that A is invertible if and only if A is bounded below and surjective. [6]

x-x-x-x-x-x