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[60] SEAT No.________

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Sardar Patel University

Mathematics

M.Sc. Semester III

Tuesday, 19 March 2019

2.00 p.m. to 5.00 p.m.

PS03CMTH21 - Real Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Which of the following functions is not integrable over $(\mathbb{N}, P(\mathbb{N}), \eta)$?

(a) $f(n) = \frac{1}{n}$ (b) $f(n) = \frac{1}{n^2}$ (c) $f(n) = \frac{1}{n^3}$ (d) $f(n) = \frac{1}{n^4}$

(2) Which of the following is not a σ- finite measure space?

(a) $(\mathbb{N}, P(\mathbb{N}), \eta)$ (b) $(\mathbb{R}, \mathfrak{M}, m)$ (c) $(\mathbb{R}, \mathscr{B}, m)$ (d) $(\mathbb{R}, P(\mathbb{R}), \eta)$

(3) Let m be the Lebesgue measure. Consider the signed measure $\nu = \delta_0 - m$ on $(\mathbb{R}, \mathfrak{M})$. Which of the following is a positive set for ν?

(a) $\mathbb{Q}$ (b) $[0,1]$ (c) $\mathbb{R} - \mathbb{Q}$ (d) $[1,2]$

(4) Let η be the counting measure and μ be a measure on $(\mathbb{R}, \mathfrak{M})$. Which of the following is true?

(a) $\eta \perp \mu$ (b) $\eta \ll \mu$ (c) $\mu \ll \eta$ (d) $\mu + \eta \ll \mu$

(5) If $f(x) = \sin x$ for all $x \in \mathbb{R}$, then

(a) $\|f\|_\infty = 1$ (b) $\|f\|_\infty = 0$ (c) $\|f\|_\infty = -1$ (d) $\|f\|_\infty = [-1,1]$

(6) Let $1 \le p < q \le \infty$. Which of the following is true?

(a) $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$ (c) $L^p([0,1]) \subset L^q([0,1])$
(b) $L^p(\mathbb{R}) \supset L^q(\mathbb{R})$ (d) $L^p([0,1]) \supset L^q([0,1])$

(7) Let η be the counting measure and δ_0 be the Dirac measure concentrated at 0. Which of the following is an outer measure on $\mathbb{R}$?

(a) m (b) $m + \delta_0$ (c) $\delta_0 - \eta$ (d) $\eta + \delta_0$

(8) Let μ^* be an outer measure on X and $E, F \subset X$. Which of the following is true?

(a) $\mu^*(E) \le \mu^*(E \cap F)$ (c) $\mu^*(E \cup F) > \mu^*(E)$
(b) $\mu^*(F \cup E) \le \mu^*(E)$ (d) $\mu^*(E \cup F) \le \mu^*(F - E) + \mu^*(E - F)$

Q.2 Attempt any ***Seven***. [14]

(a) Show that the measure space $(\mathbb{R}, \mathfrak{M}, m)$ is complete.
(b) Show that every σ- finite measure space is saturated.
(c) If f be a nonnegative measurable function on a measure space $(X, \mathscr{A}, \mu)$ and if $\int_X f d\mu = 0$, then show that $f = 0$ a.e. $[\mu]$ on X.
(d) Show that every measurable subset of a positive set is a positive set.

(P.T.O)

(e) If ν is a signed measure and μ is a measure on a measurable space $(X, \mathscr{A})$, $\nu \perp \mu$ and $\nu \ll \mu$, then show that $\nu = 0$.

(f) If f is essentially bounded, then show that f is finite a.e. $[\mu]$ on X.

(g) Let $f : \mathbb{R} \to \mathbb{R}$ be $f = \chi_{[1,2]}$. Calculate $\|f\|_p$ for all $1 \leq p \leq \infty$.

(h) Let $E \subset X$ and $\mu^*(E) = 0$. Show that E is measurable.

(i) Let μ be a measure on an algebra $\mathcal{A}$ of subsets of X, and let μ^* be the induced outer measure. If $A \subset X$ and $\epsilon > 0$, then show that there is an $\mathcal{A}_\sigma$- set E with $E \supset A$ such that $\mu^*(E) \leq \mu^*(A) + \epsilon$.

Q.3

(a) Let $(X, \mathscr{A})$ be a measurable space, and let D be a dense subset of $\mathbb{R}$. Suppose that for each $\alpha \in D$ there is an associated $B_\alpha \in \mathscr{A}$ such that $B_\alpha \subset B_{\alpha'}$ whenever $\alpha < \alpha'$. Show that there is a measurable function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on B_α^c for every $\alpha \in D$. [6]

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on X converging to a function f (pointwise) on X. Show that $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$. [6]

OR

(b) Let f be a measurable function on a measure space $(X, \mathscr{A}, \mu)$. Show that $\int_E f d\mu = 0$ for all $E \in \mathscr{A}$ if and only if $f = 0$ a.e. $[\mu]$ on X. [6]

Q.4

(c) Let ν be a signed measure on a measurable space $(X, \mathscr{A})$, and let $E \in \mathscr{A}$ with $0 < \nu(E) < \infty$. Show that E contains a positive set A with $\nu(A) > 0$. [6]

(d) Let ν be a measure and μ be a σ- finite measure on a measurable space $(X, \mathscr{A})$, and let $\nu \ll \mu$. If f is a nonnegative measurable function on X, then show that $\int_E f d\nu = \int_E f[\frac{d\nu}{d\mu}] d\mu$ for every $E \in \mathscr{A}$. [6]

OR

(d) Let ν and μ be σ- finite measures on a measurable space $(X, \mathscr{A})$. Prove that there exists a pair of measures ν_0 and ν_1 such that $\nu_0 \perp \mu$, $\nu_1 \ll \mu$ and $\nu = \nu_0 + \nu_1$. [6]

Q.5

(e) If $1 \leq p < \infty$, then show that $L^p(\mu)$ is complete. [6]

(f) Let $1 \leq p < \infty$. Let $f \in L^p(\mu)$, and let $\epsilon > 0$. Prove that there is a measurable simple function φ vanishing outside a set of finite measure such that $\|f - \varphi\|_p < \epsilon$. [6]

OR

(f) State and prove Holder's inequality. [6]

Q.6

(g) Let μ be a σ- finite measure on an algebra $\mathcal{A}$ of subsets of X. and let μ^* be the outer measure induced by μ. Show that a subset E of X is (μ^*)- measurable if and only if E can be expressed as a difference $E = A - B$, where A is an $\mathcal{A}_{\sigma\delta}$- set and $\mu^*(B) = 0$. [6]

(h) Let μ^* be an outer measure on X. If $\{E_n\}$ is a sequence of pairwise disjoint measurable subsets of X and $A \subset X$, then show that $\sum_n \mu^*(A \cap E_n) = \mu^*(A \cap (\bigcup_n E_n))$. [6]

OR

(h) Let μ^* be an outer measure on X. Show that the collection $\mathbb{B}$ of all μ^*- measurable subsets of X is a σ- algebra. [6]

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