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[59/A-8] **SEAT No.**________

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Sardar Patel University
Mathematics
M.Sc. Semester III
Tuesday, 19 March 2019
2.00 p.m. to 5.00 p.m.
PS03CMTH01 - Real Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $\mathbb{B}$ be the collection of all Borel subsets of $\mathbb{R}$, and let $\mathfrak{M}$ be the collection of all measurable subsets of $\mathbb{R}$. Which of the following is true?

(a) $\mathfrak{M} \subset \mathbb{B}$ (b) $\mathfrak{M} \supset \mathbb{B}$ (c) $\mathfrak{M} = \mathbb{B}$ (d) none of these

(2) The counting measure on $\mathbb{Z}$ fails to be .... measure.

(a) finite (b) σ-finite (c) complete (d) saturated

(3) If η is the counting and δ_0 is the point mass measure at 0, then $(\eta - \delta_0)(\mathbb{Q} \cap [-1, 1]) =$ ....

(a) 0 (b) 1 (c) 2 (d) ∞

(4) Let ν be a signed measure on $(X, \mathscr{A})$. Which of the following is not a measure?

(a) $|\nu|$ (b) ν^+ (c) ν^- (d) $-\nu^-$

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \cos x$. Then the value of $\|f\|_\infty$ is

(a) 0 (b) 1 (c) -1 (d) ∞

(6) If $f, g \in L^2(\mu)$, then fg belongs to ....

(a) $L^1(\mu)$ (b) $L^2(\mu)$ (c) $L^\infty(\mu)$ (d) $L^1(\mu) \cap L^2(\mu)$

(7) If μ^* is an outer measure on X and if $E \subsetneq F$, then which of the following is true?

(a) $\mu^*(E) \leq \mu^*(F)$ (c) $\mu^*(E) = \mu^*(F)$
(b) $\mu^*(E) < \mu^*(F)$ (d) $\mu^*(F - E) = \mu^*(F) - \mu^*(E)$

(8) Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be algebras on X. Which of the following is an algebra on X?

(a) $\mathscr{A}_1 \cap \mathscr{A}_2$ (b) $\mathscr{A}_1 \cup \mathscr{A}_2$ (c) $\mathscr{A}_1 - \mathscr{A}_2$ (d) none of these

Q.2 Attempt any ***Seven***. [14]

(a) Show that finite union of sets of finite measure is of finite measure.
(b) If a set E is measurable, then show that χ_E is measurable.
(c) Let f be a nonnegative measurable function on a measure space $(X, \mathscr{A}, \mu)$. If $\int_X f d\mu = 0$, then show that $f = 0$ a.e. $[\mu]$ on X.
(d) Let E be a positive set with respect to a signed measure ν. If F is a measurable subset of E, then show that F is a positive set.
(e) If $\{A, B\}$ and $\{A_1, B_1\}$ are Hahn decompositions of $(X, \mathscr{A}, \nu)$, then show that $A \Delta A_1$ is a null set.
(f) If α is an essential bound of f, then show that $\alpha + \epsilon$ is an essential bound of f foe all $\epsilon \geq 0$.
(g) State Riesz representation theorem.

(P.T.O.)

(h) Let μ be a measure on an algebra $\mathscr{A}$ of subsets of X. Let $\{A_i\} \subset \mathscr{A}$ and $A \in \mathscr{A}$ with $A \subset \bigcup_i A_i$. Show that $\mu(A) \leq \sum_i \mu(A_i)$.

(i) If F is a cumulative distribution of a Baire measure μ, then show that F is right continuous on $\mathbb{R}$.

Q.3

(a) Let $(X, \mathscr{A})$ be a measurable space, and let f be a nonnegative measurable function X. Show that there is an increasing sequence $\{s_n\}$ of nonnegative measurable simple functions on X converging to f (pointwise) on X. [6]

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on X converging to a function f (pointwise) on X. Show that $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$. [6]

OR

(b) Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, \mathscr{A}, \mu)$ converging to a function f (pointwise) on X. If g is integrable over X and $|f_n| \leq g$ on X for all n, then show that $\int_X f d\mu = \lim_n \int_X f_n d\mu$. [6]

Q.4

(c) Let ν be a signed measure on a measurable space $(X, \mathscr{A})$, and let $E \in \mathscr{A}$ with $0 < \nu(E) < \infty$. Show that E contains a positive set A with $\nu(A) > 0$. [6]

(d) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be an integrable function. Let $\nu : \mathscr{A} \to [-\infty, \infty]$ be $\nu(E) = \int_E f d\mu$ for all $E \in \mathscr{A}$. Show that ν is a finite signed measure and find Hahn decomposition of ν. [6]

OR

(d) Let ν be a measure and μ be a σ- finite measure on a measurable space $(X, \mathscr{A})$, and let $\nu \ll \mu$. If f is a nonnegative measurable function on X, then show that $\int_E f d\nu = \int_E f[\frac{d\nu}{d\mu}]d\mu$ for every $E \in \mathscr{A}$. [6]

Q.5

(e) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Prove that $(L^p(\mu), \|\cdot\|_p)$ is complete. [6]

(f) Let $1 \leq p < \infty$. Let $f \in L^p(\mu)$, and let $\epsilon > 0$. Prove that there is a measurable simple function φ vanishing outside a set of finite measure such that $\|f - \varphi\|_p < \epsilon$. [6]

OR

(f) Let $(X, \mathscr{A}, \mu)$ be a finite measure space, and let $1 \leq p < \infty$. Suppose that g is an integrable function on $(X, \mathscr{A}, \mu)$ satisfying $\left|\int_X g\varphi d\mu\right| \leq M\|\varphi\|_p$ for some $M > 0$ and for all measurable simple functions φ. Prove that $g \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. [6]

Q.6

(g) Let μ^* be an outer measure on X. Let $\mathbb{B}$ be the σ- algebra of measurable subsets of X. Define $\overline{\mu} : \mathbb{B} \to [0, \infty]$ by $\overline{\mu}(E) = \mu^*(E)$ for every $E \in \mathbb{B}$. Show that $\overline{\mu}$ is a complete measure on $\mathbb{B}$. [6]

(h) Let μ be a σ- finite measure on an algebra $\mathcal{A}$ of subsets of X, and let μ^* be the outer measure induced by μ. Show that a subset E of X is (μ^*)- measurable if and only if E can be expressed as a difference $E = A - B$, where A is an $\mathcal{A}_{\sigma\delta}$- set and $\mu^*(B) = 0$. [6]

OR

(h) Let μ be a measure on an algebra $\mathcal{A}$, and let μ^* be the induced outer measure. Let $\mathbb{B}$ be the σ- algebra of all (μ^*)- measurable subsets of X, and let $\mathbb{B}'$ be the smallest σ- algebra of subsets of X containing $\mathcal{A}$. If μ is σ- finite, then show that the restriction of $\overline{\mu}$ to $\mathbb{B}'$ is the unique extension of μ to $\mathbb{B}'$. [6]

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