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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - III Examination (NC)

Saturday, 21st April, 2018

PS03EMTH08, Group Theory

Time: 02:00 p.m. to 05:00 p.m. Maximum marks: 70

Note: Figures to the right indicate full marks of the respective questions. Assume standard notations wherever applicable.

Q-1 Write the question number and appropriate **option number only** for each question. [8]

(a) Let G be a finite group and $a \in G$For which of the following n, , $a^n = e$?
 (i) o(G) + o(a) (ii) $o(G) - o(a)$ (iii) $\frac{o(G)}{o(a)}$ (iv) $\frac{o(a)}{o(G)}$

(b) The orbit of 3 in $(1, 4, 3, 2)(3, 5)$ is
 (i) $\{3, 4, 1, 2\}$ (ii) $\{1, 2, 3, 4, 5\}$ (iii) $\{1, 4, 2\}$ (iv) $\{5\}$

(c) The number of conjugate classes in S_3 is ____.
 (i) 1 (ii) 2 (iii) 3 (iv) 6

(d) A group of order ____ need not be abelian.
 (i) 7 (ii) 12 (iii) 25 (iv) 35

(e) Let G be a group such that $7 \mid o(G)$. Then the number of 7-Sylow subgroups in G cannot be ____.
 (i) 1 (ii) 8 (iii) 10 (iv) 15

(f) The group ____ is simple.
 (i) $\mathbb{Z}_2$ (ii) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (iii) $\mathbb{Z}_4$ (iv) $\mathbb{Z}_2 \times \mathbb{Z}_3$

(g) The number of non-isomorphic abelian groups of order 16 is ____.
 (i) 2 (ii) 4 (iii) 5 (iv) 16

(h) The invariants of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ are
 (i) 2 (ii) 1, 1 (iii) 2, 2 (iv) 2, 1, 1

Q-2 Attempt *Any Seven* of the following: [14]

(a) Give an example of two elements in S_5 that do not commute.

(b) Show that A_n is a subgroup of S_n.

(c) Show that in a group, the relation of being "conjugate of" is an equivalence relation.

(d) Let G be an abelian group and $T : G \to G$ be defined by $T(x) = x^{-1}$. Show that T is an automorphism.

(e) State second part of Sylow's theorem.

(f) Show that a group of order 15 cannot be simple.

(g) Define external direct product of groups.

(h) Define invariants of a finite abelian group G.

(i) Determine the number of non-isomorphic abelian groups of order 216.

(P.T.O.)

Q-3 (j) Let H, K be two subgroups of a finite group G. Prove that $o(HK) = \frac{o(H)o(K)}{o(H \cap K)}$. [6]

(k) (i) For a group G and $a \in G$, show that normalizer of a in G, $N(a)$ is a subgroup of G. [2]

(ii) Show that every cycle in S_n can be written as a product of transpositions. [4]

OR

(k) State and prove Cayley's theorem. [6]

Q-4 (l) Let G be a group of order p^n for some prime p and some $n \in \mathbb{N}$. Show that $Z(G) \neq \{e\}$ and deduce that every group of order p^2 is abelian. [6]

(m) State and prove Cauchy's theorem (assume that the theorem holds in the case of finite abelian groups). [6]

OR

(m) Define *inner automorphism* of a group G and prove that the set of all inner automorphisms is a subgroup of $\text{Aut}(G)$. [6]

Q-5 (n) State and prove Sylow's theorem. [6]

(o) Prove that a group of order 225 is abelian. [6]

OR

(o) For a fixed prime p, let $n(k)$ denote the highest power of prime p which divides $(p^k)!$, Show that $n(k) = 1 + p + \cdots + p^{k-1}$. [6]

Q-6 (p) Let G be a group. Suppose that G is the internal direct product of $N_1, N_2 \ldots, N_n$ and let $T = N_1 \times N_2 \times \cdots \times N_n$. Show that G is isomorphic to T. [6]

(q) Let G be a group and $N_1, N_2, \ldots, N_n$ be normal subgroups of G such that G is the internal direct product of $N_1, N_2, \ldots, N_n$. Show that $N_i \cap N_j = \{e\}$ for $i \neq j$. Also if $a \in N_i$, $b \in N_j$ then prove that $ab = ba$. [6]

OR

(q) For an abelian group G and an integer s, let $G(s)$ be the subgroup $\{x \in G \mid x^s = e\}$. If G and G' are isomorphic, then prove that $G(s)$ and $G'(s)$ are isomorphic for every integer s. [6]

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