SEAT No.________

[76]

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SARDAR PATEL UNIVERSITY
M. Sc. (Semester III) Examination

Date: 19-4-2018 **Time: 2.00 To 5.00 p.m.**

Subject: MATHEMATICS **Paper No. PS03EMTH01 – (Functional Analysis II)**

Total Marks: 70

Note: Throughout the paper, X and Y denote nlspaces.

1. Choose the correct option for each question: [8]

(1) If $x \in K^n$, then which of the following is true?
(a) $\|x\|_1 \le \|x\|_2$ (b) $\|x\|_1 \le \|x\|_\infty$ (c) $\|x\|_1 = \|x\|_\infty$ (d) $\|x\|_2 \le \|x\|_1$

(2) Every linear functional on X is continuous, if X =
(a) l^1 (b) C[0, 1] (c) $\mathbb{C}^n$ (d) none of these

(3) Let X & Y be nlspaces. Which of the following need not be a Banach space?
(a) BL(X, K) (b) BL(X,Y) (c) BL(X′,Y) (d) none of these

(4) A map F: $(C^1[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$ defined by F(x) = x′ is
(a) closed but not continuous (c) closed and continuous
(b) continuous but not closed (d) none of these

(5) Let A ∈ BL(X). Which of the following is true?
(a) $\sigma(A) \subset \sigma_e(A)$ (b) $\sigma(A) \subset \sigma_a(A)$ (c) $\sigma_a(A) \subset \sigma_e(A)$ (d) $\sigma_e(A) \subset \sigma(A)$

(6) For x ∈ X, let j_x: X′ → K be defined by $j_x(f) = f(x)$. Then $\|j_x\| =$
(a) 1 (b) $\|f\|$ (c) $\|x\|$ (d) none of these

(7) Let F ∈ BL(X,Y). Then
(a) $\|F\| < \|F'\|$ (b) $\|F\| = \|F'\|$ (c) $\|F\| > \|F'\|$ (d) none of these

(8) In which of the following nlsaces, weak convergence and norm convergence are same?
(a) l^1 (b) l^2 (c) l^p, p > 2 (d) none of these

2. Attempt any SEVEN: [14]

(a) Let $E_1, E_2 \subset X$ and E_1 be open in X. Show that $E_1 + E_2$ is open in X.
(b) State Holder's inequality.
(c) If X is an infinite dimensional nls, then show that there is linear map from X to K which is not continuous.
(d) Show that $\mathbb{R}^2$ with $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$, is not strictly convex.
(e) If a map F: X→Y is bijective and closed, then show that F^{-1} is closed.
(f) Let X be a Banach space. If a series $\sum_n x_n$ of elements of X is absolutely summable, then show that it is summable in X.
(g) Define $\sigma_e(A)$, $\sigma_a(A)$ and $\sigma(A)$.
(h) Prove: If F ∈ BL(X,Y) and G ∈ BL(Y, Z), then (GF)′ = F′G′.
(i) Show that if $\{x_n\}$ is a sequence in X and if $x_n \xrightarrow{w} x$ & $x_n \xrightarrow{w} y$, then x = y.

(P.T.O.)

3. (a) For $x = (x(1), x(2), \ldots x(n)) \in K^n$, show that $\|x\|_p = (\sum_{i=1}^{n} |x(i)|^p)^{1/p}$ defines a norm on K^n, $(1 < p < \infty)$. [6]

(b) Let Y be a closed subspace of a nls X and $Y \neq X$. Prove that, for $0 < r < 1$, there exists $x_r \in X$ such that $\|x_r\| = 1$ and $r < \text{dist}(x_r, Y) \leq 1$. [6]

OR

(b) Let $F \in BL(X,Y)$. Define a map $\widetilde{F} : X/Z(F) \to Y$ by $\widetilde{F}(x + Z(F)) = F(x)$. Show that $\widetilde{F}$ is linear and continuous. [6]

4. (a) State and prove Hahn-Banach separation theorem. [6]

(b) Prove: A Banach space cannot have a denumerable basis. [6]

OR

(b) Prove: A subset $E \subset X$ is bounded if and only if $f(E)$ is bounded in K, for every $f \in X'$. [6]

5. (a) Prove: If X and Y are Banach spaces and $F: X \to Y$ is a closed linear map, then F is continuous. [6]

(b) Let Z be a closed subspace of a nls X. Show that the quotient map Q from X to X/Z is continuous and open. [6]

OR

(b) Prove: $k \in \sigma_a(A)$ if and only if there is a sequence $\{x_n\}$ in X such that $\|x_n\| = 1$ for all n and $\|Ax_n - kx_n\| \to 0$ as $n \to \infty$. $(A \in BL(X))$ [6]

6. (a) Let X be a finite dimensional space with $\dim X = m$ and let $\{a_1, a_2, \ldots, a_m\}$ be a basis for X. Show that $\dim X' = m$. [6]

(b) Define weak and weak* convergence in X'. Show that if $\{x_n'\}$ is a sequence in X' and $x' \in X'$, then $x_n' \xrightarrow{\|\cdot\|} x' \Rightarrow x_n' \xrightarrow{w} x' \Rightarrow x_n' \xrightarrow{w*} x'$. [6]

OR

(b) Let X be a separable nls. Prove that every bounded sequence in X' has a weak* convergent subsequence. [6]

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