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SEAT No. ________

[115] A-60

Sardar Patel University
Mathematics
M.Sc. Semester III
Tuesday, 10 April 2018
2.00 p.m. to 5.00 p.m.
PS03CMTH01 - Real Analysis II

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $\mathbb{B}$ is the collection of all Borel subsets of $\mathbb{R}$, and let $\mathcal{O}$ be the collection of all open subsets of $\mathbb{R}$. Which of the following is true?

(a) $\mathcal{O} \subset \mathbb{B}$ (b) $\mathcal{O} \supset \mathbb{B}$ (c) $\mathcal{O} = \mathbb{B}$ (d) none of these

(2) The Lebesgue measure on $\mathbb{R}$ fails to be .... measure.

(a) finite (b) σ-finite (c) complete (d) saturated

(3) If η is the counting, then $\eta(\mathbb{Q} \cap [-1,1]) =$ ....

(a) 0 (b) 1 (c) 2 (d) ∞

(4) Let ν be a signed measure and μ be a measure on $(X, \mathscr{A})$. Which of the following implies that ν is a measure?

(a) $\nu \ll \mu$ (b) $\nu \perp \mu$ (c) $\mu = 0$ (d) $\nu(E) \geq 0,\ E \in \mathscr{A}$

(5) Let $f, g \in L^\infty(\mu)$. Then $f + g$ is in

(a) $L^1(\mu)$ (b) $L^2(\mu)$ (c) $L^\infty(\mu)$ (d) none of these

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be $f = \chi_{[-1,1]}$. Then $\|f\|_1$ equals

(a) 0 (b) 1 (c) 2 (d) ∞

(7) Let η be the counting measure, δ_0 a point mass measure and m a Lebesgue measure. Which of the following is not an outer measure on $(\mathbb{R}, P(\mathbb{R}))$?

(a) η (b) δ_0 (c) $\eta + \delta_0$ (d) m

(8) Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be algebras on X. Which of the following is an algebra on X?

(a) $\mathscr{A}_1 \cap \mathscr{A}_2$ (b) $\mathscr{A}_1 \cup \mathscr{A}_2$ (c) $\mathscr{A}_1 - \mathscr{A}_2$ (d) none of these

Q.2 Attempt any ***Seven.*** [14]

(a) Is it true that every σ- finite measure space is a finite measure space? Why?

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be $f = 2\chi_{[1,2]} + 3\chi_{[5,7]}$. Evaluate $\int_{\mathbb{R}} f dm$.

(c) Give an example of a function f such that f^2 is measurable but f is not measurable.

(d) If the measures ν_1 and ν_2 are absolutely continuous with respect to the measure μ, then show that $\nu_1 + \nu_2$ is absolutely continuous with respect to μ.

(e) Is every signed measure a measure? Justify.

(f) If f is essentially bounded, then show that f is finite a.e..

(g) If $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, then show that $fg \in L^1(\mu)$.
(h) Let μ^* be an outer measure on X. If $E \subset X$ with $\mu^*(E) = 0$, then show that E is measurable.
(i) If F is a cumulative distribution of a Baire measure μ, then show that F is bounded and increasing.

Q.3

(a) Let $(X, \mathscr{A})$ be a measurable space, and let D be a dense subset of $\mathbb{R}$. Suppose that for each $\alpha \in D$ there is an associated $B_\alpha \in \mathscr{A}$ such that $B_\alpha \subset B_{\alpha'}$ whenever $\alpha < \alpha'$. Show that there is a unique measurable function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on B_α^c for every $\alpha \in D$. [6]

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let s and t be nonnegative measurable simple functions. Show that $\int_E (s+t)d\mu = \int_E s d\mu + \int_E t d\mu$ for all $E \in \mathscr{A}$. [6]

OR

(b) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be a measurable function on X. If $\int_E f d\mu = 0$ for every measurable subset E of X, then show that $f = 0$ a.e. $[\mu]$ on X. [6]

Q.4

(c) Let ν and μ be σ- finite measures on a measurable space $(X, \mathscr{A})$. Prove that there exists a unique pair of measures ν_0 and ν_1 such that $\nu_0 \perp \mu$, $\nu_1 \ll \mu$ and $\nu = \nu_1 + \nu_2$. [6].

(d) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f be an integrable function. Let $\nu : \mathscr{A} \to [-\infty, \infty]$ be $\nu(E) = \int_E f d\mu$ for all $E \in \mathscr{A}$. Show that ν is a finite signed measure and find Hahn decomposition of ν. [6]

OR

(d) Let ν be a signed measure on $(X, \mathscr{A})$. Define positive set and negative set. If E is a positive set, then show that $\nu(E) \geq 0$. Is the converse true? Justify. [6]

Q.5

(e) Let $(X, \mathscr{A}, \mu)$ be a finite measure space, let $1 \leq p < \infty$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that g is an integrable function on $(X, \mathscr{A}, \mu)$ satisfying $\left|\int_X g\varphi d\mu\right| \leq M\|\varphi\|_p$ for some $M > 0$ and for all measurable simple functions φ. Show that $g \in L^q(\mu)$. [6]

(f) Prove that $L^2(\mu)$ is complete. [6]

OR

(f) Let $(X, \mathscr{A}, \mu)$ be a σ- finite measure space. Let $1 \leq p < \infty$, and let $q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q(\mu)$, define $F_g : L^p(\mu) \to \mathbb{R}$ by $F_g(f) = \int_X fg d\mu$ for all $f \in L^p(\mu)$. Prove that F_g is a continuous linear functional on $L^p(\mu)$ and $\|F\| = \|g\|_q$. [6]

Q.6

(g) Let μ^* be an outer measure on X. Prove that the collection $\mathbb{B}$ of all μ^*- measurable subsets of X is a σ- algebra. [6]

(h) Let μ be a measure on an algebra $\mathcal{A}$ of subsets of X, and let μ^* be the outer measure induced by μ. Show that every element of $\mathcal{A}$ is μ^*- measurable. [6]

OR

(h) Let μ be a σ- finite measure on an algebra $\mathcal{A}$ of subsets of X, and let μ^* be the outer measure induced by μ. Prove that a subset E of X is μ^*- measurable if and only if E can be expressed as a difference $E = A - B$, where A is an $\mathcal{A}_{\sigma\delta}$- set and $\mu^*(B) = 0$. [6]

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