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Seat No. ______

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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Saturday, 27^{th} October, 2018

PS02CMTH24, Functional Analysis-I

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: (1) Figures to the right indicate marks of the respective question.
(2) Assume usual/standard notations wherever applicable.

Q-1 Choose the most appropriate option for each of the following questions. [08]

1. Let H be a Hilbert space and E be an orthonormal subset of H with 2018 elements. Then diam$(E) =$ ______.
   (a) 0 (b) 1 (c) 2018 (d) $\sqrt{2}$
2. Let X be an inner product space and $x, y \in X$ such that $\langle x, y\rangle = \|x\|\|y\|$. Then
   (a) x and y are orthogonal
   (b) $x = 0$ or $y = 0$
   (c) x and y are linearly dependent
   (d) x and y are linearly independent
3. Let Y be a subspace of a Hilbert space H such that $Y^{\perp} = \{0\}$. Then ______.
   (a) $\overline{Y} = H$ (b) $Y = H$ (c) $Y = \{0\}$ (d) none of these
4. If E is a singleton subset of Hilbert space H and $x \in H$, then number of best approximations from E to x is ______.
   (a) 0 (b) 1 (c) infinite (d) undetermined
5. Let H be a Hilbert space and $S, T \in BL(H)$ be self-adjoint such that $ST = TS$. Then ______ is not self-adjoint.
   (a) ST (b) $S + iT$ (c) S^2T (d) $S - T$
6. Let H be a Hilbert space and $T \in BL(H)$. Then ______ need not be true.
   (a) $\|T^*T\| = \|T\|^2$
   (b) $\|T^*\| = \|T\|$
   (c) $\|T^*T\| = \|T^*\|^2$
   (d) $\|T^2\| = \|T\|^2$
7. Let H be a Hilbert space and $T \in BL(H)$ be normal. Then $\sigma_a(T) =$ ______.
   (a) $\sigma(T)$ (b) $\sigma_e(T)$ (c) $W(T)$ (d) $\overline{W(T)}$
8. Let H be a Hilbert space and $T \in H$. If $\lambda \notin \sigma_e(T)$, then $T - \lambda I$ is ______.
   (a) one-one (b) invertible (c) onto (d) bounded below

Q-2 Attempt *any seven* of the following. [14]

(a) Define inner product on a linear space X over $K = \mathbb{R}$ or $\mathbb{C}$.

(b) State and prove Parallelogram law for an inner product space.

(c) Show that an orthonormal set of an inner product space is linearly independent.

(d) Compute the Gram matrix of $x_1 = (1, 1, 1)$, $x_2 = (-1, 1, 0)$ and $x_3 = (1, 1, -2)$.

(e) Let H be a Hilbert space and $\{x_n\}$ be a sequence in H. If $x_n \to x$, then show that $x_n \to x$ weakly.

(f) Let H be a Hilbert space and $T \in BL(H)$. Show that $\ker(T) = \ker(T^*T)$.

(g) Let H be a Hilbert space and $S \in BL(H)$ be invertible in $BL(H)$. Then show that S^* is invertible in $BL(H)$ and $(S^*)^{-1} = (S^{-1})^*$.

(PTO)

(h) Let H be a Hilbert space and $T \in BL(H)$. Show that $\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$.

(i) Show that the identity operator on an infinite-dimensional Hilbert space is not compact.

Q-3 (a) State and prove Bessel's inequality. [06]

(b) Let X be an inner product space and E be an orthonormal subset of X. Show that the set $E_x = \{u \in E : \langle x, u \rangle \neq 0\}$ is countable for each $x \in X$. [06]

OR

(b) State and prove Riesz-Fischer theorem. [06]

Q-4 (a) State and prove Riesz-representation theorem. [06]

(b) Let X and Y be normed linear spaces and $T : X \to Y$ be a linear map. Show that T is bounded if and only if T is continuous at 0. [06]

OR

(b) State and prove Projection theorem. [06]

Q-5 (a) Let H be a Hilbert space and $T \in BL(H)$. If T^* is bounded below, then show that $R(T) = H$. [06]

(b) Let H be a Hilbert space and $T \in BL(H)$. Show that there exists a unique $S \in BL(H)$ such that $\langle Tx, y \rangle = \langle x, Sy \rangle$ for every $x, y \in H$. [06]

OR

(b) Let H be a Hilbert space and $T \in BL(H)$. Show that T is unitary if and only if T is an onto isometry. [06]

Q-6 (a) Let H be a Hilbert space, $T \in BL(H)$. Show that [06]

$$\sigma(T) = \sigma_a(T) \cup \{\bar{\mu} \mid \mu \in \sigma_e(T^*)\}.$$

(b) Let H be a Hilbert space and $T \in BL(H)$. If T is compact, then show that T^* is compact. [06]

OR

(b) Define Hilbert-Schmidt operator. Let H be a separable Hilbert space and T be Hilbert-Schmidt operator on H. Show that T^* is Hilbert-Schmidt. [06]

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