[82] SEAT No.______ No of printed pages: 2

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M.Sc. (Sem-II), PS02CMTH22, Algebra-I;

Tuesday, 23^{rd} October, 2018; 10.00 a.m. to 01.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. Which of the following is a unit in the ring $\mathbb{Z}$?
   (A) 0 (B) -1 (C) 2 (D) none of these
2. Which is from the following is not an Euclidean ring ?
   (A) $\mathbb{R}[x]$ (B) $(\mathbb{Z}, +, \cdot)$ (C) $(2\mathbb{Z}, +, \cdot)$ (D) none of these
3. The number $\sqrt{1+\sqrt{2}}$ is algebraic over $\mathbb{Q}$ of degree
   (A) 1 (B) 5 (C) 2 (D) 4
4. The polynomial x^2+2 is reducible over
   (A) $\mathbb{R}$ (B) $\mathbb{Q}$ (C) $\mathbb{C}$ (D) none of these
5. $o(G(\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}))) =$
   (A) 2 (B) 1 (C) 3 (D) none of these
6. The degree of the splitting field of x^4-1 over $\mathbb{Q}$ is
   (A) 1 (B) 2 (C) 3 (D) 4
7. Which is not normal extension of $\mathbb{Q}$?
   (A) $\mathbb{Q}(\sqrt{2})$ (B) $\mathbb{Q}(\sqrt{7})$ (C) $\mathbb{Q}$ (D) none of these
8. The polynomial $x^2-5 \in \mathbb{Q}[x]$ is
   (A) reducible over $\mathbb{Q}$ (B) solvable by radicals over $\mathbb{Q}$
   (C) not solvable by radicals over $\mathbb{Q}$ (D) none of these

Q.2 Attempt any ***seven***: [14]

(a) State Pigeonhole Principle.
(b) If R is an Euclidean ring and $a, b, c \in R$ with $a \mid b, b \mid c$ then show that $a \mid c$.
(c) Find $[\mathbb{C} : \mathbb{R}]$.
(d) Is $\sin 1^0$ algebraic over $\mathbb{Q}$? Justify.
(e) Show that any two minimal, integer monic polynomials for a over F are equal.
(f) Define splitting field.
(g) Find $G(\mathbb{C} : \mathbb{R})$.
(h) Define radical extension.
(i) Define solvable group.

(P.T.O.)

Q.3

(a) Show that every Euclidean ring is a principal ideal ring and possesses a unit element. [6]

(b) If p is a prime number, prove that the polynomial $x^{p-1} + x^{p-2} + \cdots x^2 + x + 1$ is irreducible over the field of rational numbers. [6]

OR

(b) In an Euclidean ring R, show that the ideal $< a >$ is a maximal ideal in R if and only if a is a prime element of R.

Q.4

(a) If L is a finite extension of K and if K is a finite extension of F then show that L is a finite extension of F. [6]

(b) Show that a polynomial of degree n over a field can have at most n roots in any extension field. [6]

OR

(b) Construct a field containing exactly 9 elements. State results which you use.

Q.5

(a) Let $f(x) \in F[x]$. Then show that $f(x)$ has a multiple root iff $f(x)$ and $f'(x)$ have nontrivial common factor. [6]

(b) If If a and b are algebraic over F ; whose characteristic zero, then show that there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$. [6]

OR

(b) Find the degree of the splitting field of $x^4 + x^2 + 1$ over $\mathbb{Q}$.

Q.6

(a) Show that K is a normal extension of F if K is the splitting field of some polynomial over F. [6]

(b) Show that a group G is solvable iff $G^{(k)} = \{e\}$ for some $k \in \mathbb{N}$. [6]

OR

(b) State the fundamental theorem of Galois theory.

—X—