Seat No. _______

[95]

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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Saturday, 27th October, 2018

PS02CMTH04, Functional Analysis-I

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: (1) Figures to the right indicate marks of the respective question.
(2) Assume usual/standard notations wherever applicable.

Q-1 Choose the most appropriate option for each of the following questions. [08]

1. A normed linear space is a/an _______ space.
   (a) inner product (b) Hilbert (c) Banach (d) metric
2. Which of the following is not separable?
   (a) ℓ^∞ (b) $\mathbb{R}^n$ (c) ℓ^2 (d) none of these
3. Let $H = \mathbb{R}^2$ be Hilbert space and $Y = \{(0, x(2)) \in \mathbb{R}^2\} \subset H$. Then $Y^\perp =$ _______.
   (a) Y (c) $\{(x(1), 0) \in \mathbb{R}^2\}$
   (b) $\{(0,0)\}$ (d) $\{(-x(1), 0) \in \mathbb{R}^2\}$
4. Let H be a Hilbert space and $E \subset H$ be non-empty, closed, and convex. Then the number of best approximations from $x \in H$ to E is _______.
   (a) 0 (b) 1 (c) 2 (d) infinite
5. Let H be a Hilbert space and $S, T \in BL(H)$ be self-adjoint. Then _______ is self-adjoint.
   (a) S^2T (b) T^2S (c) $S-T$ (d) $S+iT$
6. Let H be a Hilbert space, $T \in BL(H)$ be such that $\ker(T^*) = \{0\}$. Then _______.
   (a) $\ker(T) = \{0\}$ (b) $R(T) = H$ (c) $R(T^*) = \{0\}$ (d) $\overline{R(T)} = H$
7. Let H be a complex Hilbert space and $T \in BL(H)$ be self-adjoint. Then the numerical range $W(T)$ is _______.
   (a) $\{0\}$ (b) the unit circle (c) a subset of $\mathbb{R}$ (d) $\mathbb{R}$
8. Let H be a Hilbert space such that identity operator on H is compact. Then $\dim(H)$ is _______.
   (a) finite (b) infinite (c) 0 (d) 1

Q-2 Attempt *any seven* of the following. [14]

(a) State Pythagoras theorem for an inner product space.

(b) Let X be a normed linear space. Show that $S_1(0) = \{x \in X : \|x\| < 1\}$ is convex.

(c) Let H be a Hilbert space and $E \subset H$. Show that $E^\perp$ is a subspace of H.

(d) Compute the Gram matrix of $x_1 = (2, -1, -1)$, $x_2 = (0, 3, -3)$ and $x_3 = (1, 1, 1)$.

(e) Let H be a Hilbert space and $T \in BL(H)$. If T is bounded below, then show that T is one-one.

(f) Let H be a Hilbert space and $T \in BL(H)$. Show that $\ker(T) = \ker(T^*T)$.

(g) Let H be a Hilbert space and $T \in BL(H)$. Show that $\|Tx\| = \|x\|$ if and only if $T^*T = I$.

(PTO)

(h) Define eigen spectrum of a bounded linear operator on a Hilbert space.

(i) Let H be a Hilbert space and $T \in BL(H)$ be normal. Show that eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

Q-3 (a) State and prove Schwarz inequality. [06]

(b) Show that the normed linear space $(\ell^p, \|\cdot\|_p)$ is an inner product space if and only if $p = 2$. [06]

OR

(b) State and prove Gram-Schmidt orthonormalization theorem. [06]

Q-4 (a) State and prove Riesz-representation theorem. [06]

(b) Let X be an inner product space, Y be a subspace of X, and $x \in X$. Show that $y \in Y$ is a best approximation from Y to x if and only if $(x - y) \perp Y$. [06]

OR

(b) State and prove unique Hahn-Banach extension theorem. [06]

Q-5 (a) Let H be a Hilbert space and $T \in BL(H)$. Show that there is a unique $S \in BL(H)$ such that $\langle Tx, y\rangle = \langle x, Sy\rangle$ for every $x, y \in H$ and $\|S\| \le \|T\|$. [06]

(b) Let H be a Hilbert space and $T \in BL(H)$. If T is self-adjoint, then show that [06]

$$\|T\| = \sup\{|\langle Tx, x\rangle| : x \in H,\ \|x\| \le 1\}.$$

OR

(b) Let H be a Hilbert space and $T \in BL(H)$. If T is onto, then show that T^* is bounded below. [06]

Q-6 (a) Let H be a Hilbert space and $T : H \to H$ be compact linear transformation. Show that T is bounded. Does the converse hold? Justify. [06]

(b) Let H be a Hilbert space, $H \neq \{0\}$ and $T \in BL(H)$. If T is self-adjoint, then show that $m_T \in \sigma(T)$, where $m_T = \inf\{\lambda : \lambda \in W(T)\}$. [06]

OR

(b) Define Hilbert-Schmidt operator. Let H be a separable Hilbert space and T be a Hilbert-Schmidt operator on H. Show that T is compact. [06]

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