[97/A11] SEAT No.________

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Sardar Patel University
Mathematics
M.Sc. Semester II
Thursday, 25 October 2018
10.00 a.m. to 01.00 p.m.
PS02CMTH03 – Differential Geometry

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) A Cartesian representation of the curve $\overline{\gamma}(t) = (\sin^2 t, \cos^2 t)$, $t \in \mathbb{R}$, is

(a) $x + y = 1$ (b) $x + y = 0$ (c) $x - y = 1$ (d) none of these

(2) Let $a, b > 0$. Then the curve $\overline{\gamma}(t) = (a \sin t, b \cos t)$, $t \in \mathbb{R}$, has infinitely many vertices if and only if

(a) $a = b$ (b) $a < b$ (c) $a > b$ (d) $b = a + 1$

(3) The equation of the tangent space to $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$ is

(a) $z = 0$ (b) $z = 1$ (c) $x = 0$ (d) $y = 0$

(4) Which of the following is not a smooth surface?

(a) $x^2 + y^2 - z^2 = 1$ (b) $x^2 + y^2 + z^2 = 1$ (c) $x^2 + y^2 = z$ (d) $x^2 + y^2 = z^2$

(5) The image of the Gauss map for a plane is

(a) sphere (b) hyperboloid (c) paraboloid (d) a point

(6) The mean curvature on a plane is

(a) -1 (b) 1 (c) $\frac{1}{2}$ (d) 0

(7) Which of the following maps preserve Gaussian curvature?

(a) local diffeomorphism
(b) conformal
(c) diffeomorphism
(d) None of these

(8) The sum of interior angle of triangle on a surface of Gaussian curvature -1 is

(a) $> \pi$ (b) $< \pi$ (c) $= \pi$ (d) 2π

Q.2 Attempt any ***Seven***. [14]

(a) Compute the arc-length of $\overline{\gamma}(t) = (e^{kt} \cos t, e^{kt} \sin t)$ starting at the point $(1, 0)$.

(b) Give an example of a plane curve having signed curvature $\kappa_s(s) = \frac{1}{1+s^2}$.

(c) Define vertex of a plane curve $\overline{\gamma}$.

(PTO)

(d) Compute first fundamental form on $\sigma(u,v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.

(e) Compute the surface area of $\{(x,y,z) : x^2+y^2 = 1, |z| \le 1\}$.

(f) Let σ be a surface patch of an oriented surface with the unit normal $\overline{\text{N}}$. Show that $\overline{\text{N}}_u \sigma_u = -L$ and $\overline{\text{N}}_u \sigma_v = -M$.

(g) Let $\overline{\gamma}$ be a unit-speed curve on an oriented surface $\mathcal{S}$. Define normal curvature and geodesic curvature of γ.

(h) Define Christoffel's symbols of second kind on a regular patch σ.

(i) State Bonnet's Theorem.

Q.3

(a) Let $\overline{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Show that the curve $\overline{\alpha} = \overline{\text{t}}$ is a regular curve. Also, find curvature and torsion of $\overline{\alpha}$. [6]

(b) Let $\overline{\gamma} : (a,b) \to \mathbb{R}^2$ be a unit-speed curve, and let $s_0 \in (a,b)$. Let $\varphi_0 \in \mathbb{R}$ such that $\dot{\overline{\gamma}}(s_0) = (\cos\varphi_0, \sin\varphi_0)$. Show that there exists a unique smooth map $\varphi : (a,b) \to \mathbb{R}$ such that $\dot{\overline{\gamma}}(s) = (\cos\varphi(s), \sin\varphi(s))$ for all $s \in (a,b)$ and $\varphi(s_0) = \varphi_0$. [6]

OR

(b) Let p and q be positive reals. Show that $\int_0^{2\pi} \sqrt{p^2\sin^2 t + q^2\cos^2 t}\, dt \ge 2\pi\sqrt{pq}$. Also, show that equality holds if and only if $p = q$. [6]

Q.4

(c) Define surface. Show that the set $\{(x,y,z) \in \mathbb{R}^3 : z = x^2 - 4y^2\}$ is a surface. [6]

(d) If a smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local isometry, then show that $\langle\cdot,\cdot\rangle_p = f^*\langle\cdot,\cdot\rangle_p$ on $T_p\mathcal{S}_1$ for all $p \in \mathcal{S}_1$. [6]

OR

(d) Let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be smooth surfaces. Prove the following statements. [6]

(A) If $p \in \mathcal{S}_1$, then the derivative at p of the identity map from $\mathcal{S}_1$ to itself is the identity map from $T_p\mathcal{S}_1$ to itself.

(B) If $f : \mathcal{S}_1 \to \mathcal{S}_2$ and $g : \mathcal{S}_2 \to \mathcal{S}_3$ are smooth maps, then for all $p \in \mathcal{S}_1$, $D_p(g \circ f) = D_{f(p)}g \circ D_p f$.

Q.5

(e) Compute the Gaussian curvature and mean curvature of $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, where $(\frac{df}{du})^2 + (\frac{dg}{du})^2 = 1$. [6]

(f) Compute the principal curvatures and the principal vectors of $\sigma(u,v) = (u, v, u^2+v^2)$ at $(0,0,0)$. [6] [6]

OR

(f) Show that the principal curvatures at a point of a surface are maximum and minimum values of the normal curvatures. Moreover, the principal vectors (directions) are the directions giving these maximum and minimum values. [6]

Q.6

(g) If $\overline{\gamma}$ is a geodesic on the sphere $x^2+y^2+z^2 = 1$, then show that $\overline{\gamma}$ is part of a great circle. [6]

(h) Let σ be a surface patch of an oriented surface $\mathcal{S}$. Show that $L_v - M_u = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}$ and $M_v - N_u = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}$. [6]

OR

(h) If σ is a regular patch of an oriented surface, then show that $\sigma_{uu}\sigma_u = \frac{1}{2}E_u$, $\sigma_{uu}\sigma_v = F_u - \frac{1}{2}E_v$, $\sigma_{uv}\sigma_u = \frac{1}{2}F_v$, $\sigma_{uv}\sigma_v = \frac{1}{2}G_u$, $\sigma_{vv}\sigma_u = F_v - \frac{1}{2}G_u$ and $\sigma_{vv}\sigma_v = \frac{1}{2}G_v$. [6]

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