Sc

(32 & A-7) **SEAT No.**_______

No of printed pages: 2

**Sardar Patel University**

Mathematics

M.Sc. Semester II

Saturday, 20 October 2018

10.00 a.m. to 01.00 p.m.

PS02CMTH01 - Real Analysis I

**Maximum Marks: 70**

Q.1 Fill in the blanks. [8]

(1) $m^*([0,2] \cap (\mathbb{R} - \mathbb{Q})) = \ldots\ldots$

(a) 0 (b) 1 (c) 2 (d) $\infty$

(2) Let $E$ be a nonmeasurable subset of $\mathbb{R}$. Which of the following is true?

(a) $m^*E > 0$ (b) $m^*E = 0$ (c) $m^*E < 0$ (d) none of these

(3) Which of the following functions is not Riemann integrable over $[0,1]$?

(a) $x^2 \sin \frac{1}{x}$ (b) $\sin^2 x$ (c) $x^2 \cos \frac{1}{x}$ (d) $\frac{1}{x^2}$

(4) Let $f$ be a bounded measurable function vanishing outside a set of finite measure, and let $E$ and $F$ be measurable subsets of $\mathbb{R}$ with $E \subset F$. Which of the following is true?

(a) $\int_E f \leq \int_F f$ (b) $\int_E f \geq \int_F f$ (c) $\int_E f = \int_F f$ (d) none of these

(5) Let $E$ be a measurable subset of $\mathbb{R}$. Which of the following implies that $\chi_E$ is not integrable over $\mathbb{R}$?

(a) $mE = 0$ (b) $mE = 1$ (c) $0 < mE < \infty$ (d) $mE = \infty$

(6) Let $f$ and $g$ be integrable over $E$. Which of the following may not be integrable over $E$?

(a) $f + g$ (b) $f - g$ (c) $2f + 3g$ (d) $f - g^2$

(7) Let $f(x) = x^3$ for all $x \in [0,1]$. Then the total variation of $f$ over $[0,1]$ is

(a) 0 (b) 1 (c) 2 (d) 4

(8) Let $f : [a,b] \to \mathbb{R}$ be of bounded variation and $\alpha \in \mathbb{R}$. Which of the following is true?

(a) $T_a^b(\alpha f) = \alpha T_a^b(f)$ (c) $T_a^b(f) < 0$

(b) $T_a^b(\alpha^3 f) = \alpha^3 T_a^b(f)$ (d) $T_a^b(\alpha^2 f) = \alpha^2 T_a^b(f)$

Q.2 Attempt any ***Seven***. [14]

(a) Define $F_\sigma$ and $G_\delta$ subsets of $\mathbb{R}$.

(b) Define $\sigma$- algebra on a set $X$.

(c) If $f^2$ is measurable over $\mathbb{R}$, then show that $f$ need not be measurable.

(d) If $\varphi$ is a nonnegative measurable simple function vanishing outside a set of finite measure, then show that $\int \varphi \geq 0$.



(P.T.O.)

(e) If $f$ is nonnegative measurable function on $E$ and $\int_E f = 0$, then show that $f = 0$ a.e. in $E$.
(f) If $f$ is integrable over $E$, then show that $|\int_E f| \leq \int_E |f|$.
(g) State Fatou's Lemma.
(h) Let $f : [0,1] \to \mathbb{R}$ be $f(x) = x$. Show that $f$ is absolutely continuous.
(i) If $f : [a,b] \to \mathbb{R}$ is a difference of two increasing real valued functions, then show that $f$ is of bounded variation.

Q.3

(a) If $a \in \mathbb{R}$, then show that the interval $(a, \infty)$ is measurable. [6]
(b) Show that $[0,1]$ contains a nonmeasurable set. [6]

**OR**

(b) Let $\{f_n\}$ be a sequence of measurable functions on a measurable set $E$. Show that $\liminf_n f_n$ and $\limsup_n f_n$ are measurable functions. [6]

Q.4

(c) If $\{f_n\}$ is a sequence of measurable functions that converge to a real valued function $f$ a.e. on a measurable set $E$ of finite measure, then show that given $\eta > 0$, there is a measurable subset $A \subset E$ with $mA < \eta$ such that $\{f_n\}$ converges to $f$ uniformly on $E - A$. [6]
(d) Let $f$ be defined and bounded on a measurable set $E$ with $mE$ finite. Suppose that $\inf_{f \leq \psi} \int_E \psi(x)dx = \sup_{f \geq \varphi} \int_E \varphi(x)dx$, where $\varphi$ and $\psi$ are simple functions. Show that $f$ is measurable. [6]

**OR**

(d) Let $\{f_n\}$ be a sequence of measurable functions defined on a set $E$ of finite measure, and let $|f_n(x)| \leq M$ for all $x \in E$ and for all $n \in \mathbb{N}$ for some $M > 0$. If $f(x) = \lim_n f_n(x)$ for each $x$ in $E$, then show that $\int_E f = \lim_n \int_E f_n$. [6]

Q.5

(e) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to $f$ on $E$. Show that there is a subsequence $\{f_{n_k}\}$ that converges to $f$ almost everywhere on $E$. [6]
(f) Let $f$ be a nonnegative measurable function defined on a measurable set $E$, and let $\{E_n\}$ be a disjoint sequence of measurable sets with $\bigcup_n E_n = E$. Show that $\int_E f = \sum_n \int_{E_n} f$. State the results you use. [6]

**OR**

(f) Let $f$ be integrable over $E$. Show that for given $\epsilon > 0$, there is $\delta > 0$ such that $|\int_F f| < \epsilon$ whenever $F$ is a measurable subset of $E$ with $mF < \delta$. [6]

Q.6

(g) Let $f$ be an integrable function on $[a,b]$, and let $F(x) = F(a) + \int_a^x f$ for all $x \in [a,b]$. Show that $F'(x) = f(x)$ for almost all $x$ in $[a,b]$. [6]
(h) If $f$ is integrable on $[a,b]$ and $\int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then show that $f = 0$ a.e. in $[a,b]$. [6]

**OR**

(h) Show that a function $F$ on $[a,b]$ is an indefinite integral if and only if it is absolutely continuous. [6]

ɧɧɧɧɧɧɧɧɧ

— x —