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Sardar Patel University

M. Sc. *(Second Semester)* Examination

Tuesday, 25th October 2016

Course No. PS02CMTH04 : Functional Analysis – I

Time: 10.00 a.m. to 01.00 p.m. *Maximum marks: 70*

Note: Figures to the right indicates marks. K denotes the field R or C.

1. Fill up the gaps in the following from the given option : [8]
   i) Let H be a Hilbert space and x, y ∈ H be orthonormal. Then $\|x + y\|$ = _____.
   (a) 1 (b) 2 (c) 0 (d) $\sqrt{2}$
   ii) For $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$, define $<x, y>$ = __________ . Then $<.,.>$ is not an inner product on R^2.
   (a) $x_1y_1 + x_2y_2$ (b) $x_1x_2 + y_1y_2$
   (c) $5x_1y_1 + 6x_2y_2$ (d) $6x_1y_1 + 5x_2y_2$
   iii) Let X be an inner product space and $\{u_n\}$ be a sequence of orthonormal elements in X. Then __________.
   (a) $\|u_n\| \to 0$ (b) $u_n \to 0$
   (c) $u_n \to 0$ weakly (d) $\{u_n\}$ is Cauchy
   iv) Let X be an inner product space, $E \subset X$ and $x \in \bar{E}$. If there is a best approximation form E to x, then __________.
   (a) $x = 0$ (b) $x \in E^c$ (c) $x \in E^\circ$ (d) $x \in E$
   v) Let H be a Hilbert space and T ∈ BL(H) be such that $\|Tx\| = \|x\|$ for each x ∈ H. Then __________.
   (a) T is onto (b) $T^*T = I$ (c) $TT^* = I$ (d) T is not one to one
   vi) Let H be a Hilbert space and S, T ∈ BL(H) be self-adjoint. Then _______ self-adjoint.
   (a) $3S + 4T$ (b) ST (c) $3S + 4iT$ (d) $3iS + 4T$
   vii) Let H be a Hilbert space and T ∈ BL(H). Then __________.
   (a) $\ker(T^*) = \ker(T^*T)$ (b) $\ker(T) = \ker(T^*T)$
   (c) $R(T) = R(T^*T)$ (d) $R(T^*) = R(T^*T)$
   viii) Let H be a Hilbert space; T ∈ BL(H) be non-zero compact self-adjoint and $0 \neq \lambda \in \sigma(T)$. Then _________ .
   (a) $\lambda \in \sigma_e(T)$ (b) $T-\lambda I$ is onto
   (c) $T-\lambda I$ is one to one (d) $T-\lambda I$ is invertible
2. Answer any SEVEN of the following: [14]
   i) Let H be a Hilbert space and T ∈ BL(H) be one to one. Define $<.,.>_T$ by $<x, y>_T = <Tx, Ty>$, x, y ∈ H. Show that $<.,.>_T$ is an inner product on H.
   ii) State and prove Pythagoras Theorem in an Inner Product Space.
   iii) Show that every strongly convergent sequence in a Hilbert space is weakly convergent.
   iv) Calculate **Gram matrix** for $x_1 = (4, 0, 0)$, $x_2 = (0, 3, 2)$, $x_3 = (4, 3, 2)$.
   v) Define $T : K^2 \to K^2$ by $T(x(1), x(2)) = (x(1) + 2x(2), 3(1) + x(2))$, $(x(1), x(2)) \in K^2$. Find T^*.
   vi) Let H be a Hilbert space and T ∈ BL(H). Show that T is one to one iff $R(T^*)$ is dense in H.
   vii) Let H be a Hilbert space and T ∈ BL(H) be bounded below. Show that there is a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ for each n and $Tx_n \to 0$.
   viii) Let H be a Hilbert space and T ∈ BL(H) be normal. Show that $\|T^*x\| = \|Tx\|$ for all x ∈ H.
   ix) Let H be a Hilbert space and S, T ∈ BL(H) be such that S is compact. Show that ST is compact.

3. a) State and prove Schwarz inequality in an inner product space. [6]

b) Let X be an infinite dimensional inner product space. If X is separable, prove that it has a countable orthonormal basis. [6]

OR

b) Let X be an inner product space and E be an orthonormal subset of X. Show that for every $x \in X$, the set $E_x = \{ u \in E : \langle u, x \rangle \neq 0 \}$ is countable. [6]

4. a) State and prove Riesz representation theorem. [6]

b) State and prove unique Hahn Banach extension theorem. [6]

OR

b) Let H be a Hilbert space; E be a closed convex subset of H and $x \in H$. Show that there is a unique best approximation y from E to x. [6]

5. a) Let H be a Hilbert space and $T^* \in BL(H)$ be bounded below. Show that $R(T) = H$. [6]

b) Define: unitary operator. Let H be Hilbert space; $A, B \in BL(H)$ be self-adjoint. and $T=A+iB$. Show that T is unitary iff $AB = BA$ and $A^2 + B^2 = I$. [6]

OR

b) Let H be a Hilbert space and $T \in BL(H)$. Define the adjoint T^* of T. [6]
If $S, T \in BL(H)$, show that $(ST)^* = T^*S^*$, $(S+T)^* = S^*+T^*$ and $\|T^*T\| = \|T\| = \|T^*\|$.

6. a) Let H be a Hilbert space and $T \in BL(H)$. Show that [6]
$\sigma(T) = \sigma_a(T) \cup \{\mu : \bar{\mu} \in \sigma_e(T^*)\}$

b) Let H be a Hilbert space and $\{T_n\}$ be a sequence of compact operators such that $T_n \to T$ in $BL(H)$. Show that T is compact. [6]

OR

b) Define Hilbert – Schmidt operator. Show that every Hilbert –Schmidt operator is compact. [6]