No of printed pages: 2

Seat No.____

[17/A-1]

Sardar Patel University

M.Sc. Semester II Examination

2016

Saturday, 29 October

10.00 to 1.00

Mathematics: PS02CMTH02

(Algebra I)

Maximum Marks: 70

Q.1 Write the correct **option number only** for each question. [8]

(*a*) _____ has exactly 4 invertible elements.

(i) $\mathbb{Z}$ (ii) $\mathbb{Z}_7$ (iii) $\mathbb{Z}_6[x]$ (iv) $\mathbb{Z}_8[x]$

(*b*) _____ is not an integral domain.

(i) $4\mathbb{Z}$ (ii) $\mathbb{Z}$ (iii) $\mathbb{Z}_4$ (iv) $\mathbb{Q}$

(*c*) $x^2 - 2$ is irreducible over _____

(i) $\mathbb{Q}$ (ii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (iii) $\mathbb{C}$ (iv) $\mathbb{R}$

(*d*) The field $\mathbb{R}[x]/\langle x^2+1\rangle$ is isomorphic to _____.

(i) $\mathbb{Q}$ (ii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ (iii) $\mathbb{C}$ (iv) $\mathbb{R}$

(*e*) $x^5 + x^2 + x + 1 = (x^3 + x + 1)(x^2 + 1)$ is true in _____.

(i) no field (ii) $\mathbb{Z}_3[x]$ (iii) $\mathbb{Z}_2[x]$ (iv) $\mathbb{Z}[x]$

(*f*) $[\mathbb{C} : \mathbb{R}] =$_____.

(i) 2 (ii) 3 (iii) 4 (iv) ∞

(*g*) The set of all real numbers, which are algebraic over $\mathbb{Q}$ is _____.

(i) countable (ii) finite (iii) uncountable (iv) empty

(*h*) The degree of the splitting field of $x^3 - 1 \in \mathbb{Q}$ over $\mathbb{Q}$ is _____

(i) 1 (ii) 2 (iii) 3 (iv) 4

Q.2 Attempt any **Seven. (Start a new page.)** [14]

(*a*) Define a *Euclidean ring* and give one example of the same.

(*b*) Define the term: *associates* and give an example of the same.

(*c*) State Eisenstein theorem.

(*d*) For $a, b \in \mathbb{Q}$, when is $a + b\sqrt{2}$ invertible in $\mathbb{Q}(\sqrt{2})$? In that case, find $c, d \in \mathbb{Q}$ such that $c + d\sqrt{2} = (a + b\sqrt{2})^{-1}$.

(*e*) Find the quotient and remainder when $x^4 + 3x^3 + 2x^2 + x + 1$ is divided by $x^2 - 2x - 1$ in $\mathbb{Z}_5[x]$.

(*f*) Define the terms: *extension field* and *algebraic extension*.

(*g*) Show that π is algebraic over $\mathbb{R}$.

(*h*) Define the term: *solvable group* and give one example of the same.

(*i*) Define the term: *symmetric rational function*.

[Contd...]

Q.3 (Start a new page.)

(a) If a, b are elements of a Euclidean ring, then show that their gcd exists. [6]

(b) If π is a prime element in the Euclidean ring $\mathcal{R}$ and $\pi \mid ab$, where $a, b \in \mathcal{R}$, then show that $\pi \mid a$ or $\pi \mid b$. [6]

OR

(b) If p is a prime integer of the form $4n+1$, then show that $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$. [6]

Q.4 (Start a new page.)

(c) Show that product of two primitive polynomials is a primitive polynomial. [6]

(d) Find all irreducible polynomials of degree less or equal to 3 in $\mathbb{Z}_2[x]$. [6]

OR

(d) Show that $x^5 + 2x + 4$ is irreducible in $\mathbb{Q}[x]$. [6]

Q.5 (Start a new page.)

(e) Prove the existence of a real number, which is not algebraic over $\mathbb{Q}$. [6]

(f) If K is an extension field of a field F and $a \in K$, then describe the internal construction of $F(a)$. [6]

OR

(f) Let F be a field and $f(x) \in F[x]$ be nonconstant. Prove the existence of a field containing at least one root of $f(x)$ [6]

Q.6 (Start a new page.)

(g) Let K be a normal extension of F, H be a subgroup of $G(K, F)$ and K_H be a fixed field of H. Then show that (i) $[K : K_H] = o(H)$ and (ii) $G(K, K_H) = H$. [6]

(h) If K is a finite extension of F, then show that $o(G(K, F)) \leq [K : F]$ [6]

OR

(h) In usual notations prove that $[F(x_1, x_2, \ldots, x_n) : S] = n!$. [6]

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