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Seat No. _______

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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Tuesday, 26th March, 2019

PS02CMTH24, Functional Analysis - I

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: (1) Figures to the right indicate marks of the respective question.
(2) Here H denotes the Hilbert space over the field K, where K is $\mathbb{R}$ or $\mathbb{C}$, and I denotes identity operator. Assume other usual/standard notations wherever applicable.

Q-1 Write the most appropriate option only for each of the following questions. [08

1. _______ space is _______ space.
   (a) A topological, a metric (b) A metric, a normed linear (c) A normed linear, an inner product (d) An inner product, a metric
2. _______ is not an inner product space.
   (a) $\mathbb{R}^6$ (b) c_{00} (c) ℓ^∞ (d) $C[0,1]$
3. $(1,0,3)$ is not a best approximation from $\mathbb{R} \times \{0\} \times \mathbb{R}$ to _______.
   (a) $(0,-2,0)$ (b) $(1,2,3)$ (c) $(1,-1,3)$ (d) $(1,0,3)$
4. If $\{u_n\}$ is an orthonormal basis of an infinite-dimensional Hilbert space, then _____.
   (a) $\|u_n\| \to \sqrt{2}$ (b) $u_n \to 0$ (c) $u_n \xrightarrow{w} 0$ (d) $\{u_n\}$ is Cauchy
5. If S is self-adjoint, $\alpha \in \mathbb{C}$ with imaginary part $\operatorname{Im}\alpha$, then ______ is self-adjoint.
   (a) $i\alpha S$ (b) $(\operatorname{Im}\alpha)S$ (c) $\bar{\alpha}S$ (d) $-i(\operatorname{Im}\alpha)S$
6. If $T \in BL(H)$ is bounded below, then _______.
   (a) T is regular (b) T^* is one-one (c) T is onto (d) T^* is onto
7. Let $T \in BL(\mathbb{C}^5)$ be a projection. Then $\sigma_a(T) =$_______.
   (a) $\{0,1\}$ (b) $\mathbb{R}$ (c) $\{1,-1\}$ (d) $\emptyset$
8. Let H be a Hilbert space and $T \in BL(H)$. If $\lambda \notin W(T)$, then _______.
   (a) $\lambda \notin \sigma_e(T)$ (b) $\lambda \notin \sigma_a(T)$ (c) $\lambda \notin \sigma(T)$ (d) $\lambda \in \sigma_e(T)$

Q-2 Attempt *any seven* of the following. [14]

(a) Let X be an inner product space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, then show that $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

(b) State and prove Parallelogram law for an inner product space.

(c) Define uniformly convex normed linear space

(d) State projection theorem.

(e) Show that a bounded subset E of a Hilbert space H is weakly bounded.

(f) Let H be a Hilbert space and $S, T \in BL(H)$. Show that $(S+T)^* = S^* + T^*$.

(g) Let H be a Hilbert space and $T \in BL(H)$ be isometry. Show that $T^*T = I$.

(h) Give an example of a Hilbert-Schmidt operator on a separable Hilbert space.

(i) Let H be a Hilbert space. For $T \in BL(H)$ show that $\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$.

(P.T.O.)

Q-3 (a) State and prove Schwarz inequality. When does the equality hold? Justify. [06]

(b) Let X be an inner product space and $\{u_1, u_2, \ldots\}$ be a countable orthonormal subset of X. Then prove that for each $x \in X$, [06]

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \text{ if and only if } x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n.$$

OR

(b) Let X be a normed linear space over K where the norm satisfies parallelogram law. For all $x, y \in X$, define $\langle \cdot, \cdot \rangle : X \to K$ by [06]

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

Prove that $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$.

Q-4 (a) Let H be a Hilbert space over K. Prove that $f : H \to K$ is continuous linear if and only if there exists $y \in H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$. [06]

(b) Let X be an inner product space, Y be a subspace of X and $x \in X$. Prove that $y \in Y$ is a best approximation from Y to x if and only if $(x - y) \perp Y$. [06]

OR

(b) Let H be a Hilbert space. Show that every bounded sequence in H has a weakly convergent subsequence. [06]

Q-5 (a) Let H be a Hilbert space and $T \in BL(H)$. Prove that there exists a unique $S \in BL(H)$ such that $\langle Tx, y \rangle = \langle x, Sy \rangle$ for every $x, y \in H$. [06]

(b) Let H be a Hilbert space and $T \in BL(H)$ be self-adjoint. Prove that $\langle Tx, x \rangle = 0$ for all $x \in H$ if and only if $T = 0$. [06]

OR

(b) Give an example with proper verification of each of the following. [06]

1. A unitary operator which is not self-adjoint.
2. A self-adjoint operator which is not unitary.

Q-6 (a) Let $T \in BL(H)$. Prove that $\sigma(T) = \sigma_a(T) \cup \{\bar{\mu} \mid \mu \in \sigma_e(T^*)\}$. [06]

(b) Let H be a Hilbert space and $T : H \to H$ be linear. Prove that if T is compact, then it is bounded. Does the converse hold? Justify. [06]

OR

(b) Let H be a Hilbert space and $T \in BL(H)$. Prove that T is compact if and only if T^* is compact. [06]

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