[54] SEAT No.________ No. of Printed Pages : 3

Sc

Sardar Patel University

M.Sc. (Mathematics) (Semester-II); Examination 2019

PS02CMTH23: Differential Geometry

Date: 23rd March, 2019 Full Marks: 70

Saturday Time: 10:00 am to 01:00 pm

Instructions:

1. **Attempt all questions.**
2. **Assume usual/standard notations wherever applicable.**
3. **Figures to the right indicate full marks.**

Q- 1 Choose the most appropriate option for each of following question: **[8]**

(i) A parametrization of a parabola $y = x^2$ is

(a) $(t^4, t^8),\ t \in R$ (b) $(t^2, t^4),\ t \in R$ (c) $(\sqrt{t}, t),\ t \in R$ (d) $(t, t^2),\ t \in R$

(ii) Equality holds in Isoperimetric Inequality if and only if

(a) $\bar{\gamma}$ is a Parabola (b) $\bar{\gamma}$ is an Ellipse (c) $\bar{\gamma}$ is a Hyperbola (d) $\bar{\gamma}$ is a Circle

(iii) A surface patch of Cylinder $S = \{(x, y, z) \in R^3 : x^2 + y^2 = 1\}$ is given by

(a) $\sigma(u,v) = (u, v, u^2 + v^2)$ (b) $\sigma(u,v) = (\cos u, \sin u, v)$

(c) $\sigma(u,v) = (\cos u, \sin v, v)$ (d) $\sigma(u,v) = (\cos v, \sin u, v)$

(iv) If a local isometry is diffeomorphism itself called

(a) a local isometry (b) an isometry (c) homeomorphism (d) conformal

(v) The second fundamental form at p is a

(a) bilinear map (b) linear map (c) nonlinear map (d) Gauss map

(vi) $z = \frac{1}{2}(k_1 x^2 + k_2 y^2)$ where k_1 and k_2 are principal curvatures then this equation represents Elliptic paraboloid if

(a) $k_1 = k_2 = 0$ (b) $k_1, k_2 < 0$ (c) $k_1, k_2 > 0$ (d) $k_1 > 0,\ k_2 < 0$

(vii) $2\sigma_{uu}\sigma_u =$

(a) G_u (b) F_u (c) E_u (d) E_v

(viii) The sum of the interior angles of a Triangle on a Sphere is

(a) $< \pi$ (b) 2π (c) $= \pi$ (d) $> \pi$

(1) (P.T.O)

Q- 2 Attempt **any Seven** **[14]**

(a) Prove that if $\bar{\gamma}$ is a unit-speed curve then $\dot{\bar{\gamma}} \perp \ddot{\bar{\gamma}}$

(b) State Frenet-Serret theorem.

(c) Prove that composition of two conformal map is a conformal map.

(d) Show that Gaussian Curvature is a square of geometric mean of Principal curvatures

(e) Find the image of the Gauss map for $\sigma(u,v) = (u, v, u^2 + v^2);\ u, v \in R$

(f) Prove that any Geodesic is unit-speed

(g) Show that a unit-speed curve on a surface S is a geodesic if its geodesic curvature is zero and hence any part of a line on a surface is a geodesic.

(h) State Gauss-Bonnet theorem.

(i) Show that $\|\dot{\bar{\gamma}}\|$ may be expressed as $\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}$

Q-3 (a) Let $\bar{\gamma} : (a,b) \to R^2$ be a unit-speed curve and let $s_0 \in (a,b)$. Let $\varphi_0 \in R$ be such that $\dot{\bar{\gamma}}(s_0) = (\cos\varphi_0, \sin\varphi_0)$ then there exists a smooth map $\varphi : (a,b) \to R$ such that $\dot{\bar{\gamma}}(s) = (\cos\varphi(s), \sin\varphi(s))$ for all $s \in (a,b)$ and $\varphi(s_0) = \varphi_0$ **[6]**

(b) State and prove Isoperimetric Inequality **[6]**

OR

(b) If $k : (a,b) \to R$ is a smooth map, then there is a unit-speed curve $\bar{\gamma} : (a,b) \to R^2$ whose signed curvature is *k*. Moreover, if $\tilde{\gamma} : (a,b) \to R^2$ is a unit-speed curve whose signed curvature is *k* then there is a direct isometry *M* of R^2 such that $\tilde{\gamma} = M \circ \bar{\gamma}$

Q-4 (a) Define Stereographic projection and hence obtain two feasible surface patches of $S = \{(x,y,z) \in R^3 : x^2 + y^2 + z^2 = 1\}$. **[6]**

(b) State and prove Necessary and Sufficient condition for the local diffeomorphism $f : S_1 \to S_2$ to be a conformal map, where S_1 and S_2 are smooth surfaces. **[6]**

OR

(b) State and prove chain rule for smooth surfaces. Let $f : S_1 \to S_2$ be a local diffeomorphism and let $\bar{\gamma}$ be a regular curve in S_1 then show that $f \circ \bar{\gamma}$ is a regular curve in S_2

Q-5 (a) Let $\sigma : U \to R^3$ be a regular surface patch and let $(u_0, v_0) \in U$. For $\delta > 0$ let $B_\delta = \{(u,v) : (u-u_0)^2 + (v-v_0)^2 < \delta^2\}$. Let $K > 0$ be the Gaussian curvature of the surface at *p* then prove that $\lim_{\delta \to 0} \dfrac{A_N(B_\delta)}{A_\sigma(B_\delta)} = |K|$. **[6]**

(2)

(b) If $\bar{\gamma}$ is a unit-speed curve on an oriented surface *S*, then its normal curvature is given by $\kappa_n = II_\gamma\langle\dot{\bar{\gamma}}, \dot{\bar{\gamma}}\rangle = \langle W_{\bar{\gamma}}(\dot{\bar{\gamma}}), \dot{\bar{\gamma}}\rangle_{\bar{\gamma}}$. If σ is a surface patch of a surface *S* and if $\bar{\gamma}(t) = \sigma(u(t), v(t))$ is a unit-speed curve on σ, then $\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$ **[6]**

OR

(b) Let σ is a surface patch of an oriented surface *S* containing a point $p = \sigma(u, v)$ then

$$II_p\langle w, x\rangle = Ldu(w)du(x) + M\{du(w)dv(x) + du(x)dv(w)\} + Ndv(w)dv(x)$$

for all $w, x \in T_p(S)$ where $du, dv : T_pS \to R$

Q-6 (a) Discuss geodesic on the unit sphere $S = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1\}$ **[6]**

(b) Define Christoffel symbol of 2[nd] kind and hence prove Codazzi-Mainardi Equations. **[6]**

OR

(b) Define Gaussian and Mean Curvatures. Let $\sigma : U \to R^3$ be a surface patch of an oriented surface *S* and let $p = \sigma(u, v)$ for some $(u, v) \in U$ then the matrix of W_p with respect to the basis $\{\sigma_u, \sigma_v\}$ of $T_p(S)$ is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$

and hence obtain $K_p = \dfrac{LN - M^2}{EG - F^2}$ and $H_p = \dfrac{LG - 2MF + NE}{2(EG - F^2)}$

—X—

(3)