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[51] **SEAT No.________**

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Sardar Patel University

M.Sc. (Sem-II), PS02CMTH22, Algebra-I;

Wednesday, 20^{th} March, 2019; 10.00 a.m. to 01.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. Which is from the following is not an Euclidean ring ?
   (A) $(2\mathbb{Z},+,\cdot)$ (B) $(\mathbb{Z},+,\cdot)$ (C) $\mathbb{R}[x]$ (D) $\mathbb{Q}[x]$
2. $\mathbb{Z}_3[x]/ < x^2+1 >$ is
   (A) an infinite field. (B) a field containing 9 elements.
   (C) a field containing 8 elements. (D) NOT a field
3. The real number π is algebraic over
   (A) $\mathbb{Q}$ (B) $\mathbb{Q}(\sqrt{2})$ (C) $\mathbb{Q}(\pi)$ (D) $\mathbb{Q}(e)$
4. The number $\sqrt{3}$ is algebraic over $\mathbb{Q}$ of degree
   (A) 2 (B) 3 (C) 1 (D) 4
5. $o\big(G(\mathbb{Q}(\sqrt{2}),\mathbb{Q})\big) =$
   (A) 1 (B) 2 (C) 3 (D) 4
6. Let $f(x)\in\mathbb{Z}_2[x]$ with $f'(x)=0$. Then there exists $g(x)\in\mathbb{Z}_2[x]$ such that $f(x)=$
   (A) $g(x^4)$ (B) $g(x)$ (C) $g(x^2)$ (D) $g(x^3)$
7. Which is not normal extension of $\mathbb{Q}$?
   (A) $\mathbb{Q}(\sqrt{2})$ (B) $\mathbb{Q}(\sqrt{5})$ (C) $\mathbb{Q}$ (D) $\mathbb{Q}(\pi)$
8. The polynomial $x^2-3 \in \mathbb{Q}[x]$ is
   (A) solvable by radicals. (B) reducible over $\mathbb{Q}$.
   (C) not solvable by radicals. (D) none of these

Q.2 Attempt any ***seven***: [14]

(a) Show that every field is Euclidean ring.
(b) Let $f(x)\in F[x]$ and $a\in F\setminus\{0\}$. If $f(ax)$ is irreducible over F then show that $f(x)$ is irreducible over F.
(c) Is x^2+x+1 irreducible over $\mathbb{Z}_2$? Justify.
(d) Is $\cos(2019°)$ algebraic over $\mathbb{Q}$? Justify.
(e) Show that $\mathbb{C}$ is algebraic extension of $\mathbb{R}$.
(f) Find $G(\mathbb{C},\mathbb{R})$.
(g) Define simple extension and give one example of it.
(h) Define radical extension.
(i) State Able's theorem.

(1) (P.T.O.)

Q.3

(a) Show that every Euclidean ring is a principal ideal ring and possesses a unit element. [6]

(b) In an Euclidean ring, show that any two nonzero elements have the least common multiple. [6]

OR

(b) Prove that $1+x+x^2+\cdots+x^{p-1}$ is irreducible over $\mathbb{Q}$, where $p > 2$ is a prime number.

Q.4

(a) If L is a finite extension of K and if K is a finite extension of F then show that L is a finite extension of F. [6]

(b) Let $p(x)$ be nonconstant polynomial of degree n over a field F. Then show that there exists an extension E of F having $[E : F] \leq n!$ such that $p(x)$ has n roots in E. [6]

OR

(b) If K is an extension of F and $a, b \in K \setminus \{0\}$ are algebraic over F, then show that ab^{-1} is algebraic over F. Does the converse hold ? Justify.

Q.5

(a) If $[K : F] < \infty$, then show that $o(G(K, F)) \leq [K : F]$. State results which you use. [6]

(b) If a and b are algebraic over F ; whose characteristic zero, then show that there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$. [6]

OR

(b) Find the degree of the splitting field of $x^p - 1$ over $\mathbb{Q}$, where $p > 2$ is a prime number.

Q.6

(a) Show that K is a normal extension of F, if K is the splitting field of some polynomial over F. [6]

(b) Show that a group G is solvable iff $G^{(k)} = \{e\}$ for some $k \in \mathbb{N}$. [6]

OR

(b) Find the Galois group of the polynomial $x^5 - 1$ over $\mathbb{Q}$.

—X—

(2)