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[53] **SEAT No.________** No of printed pages: **2**

Sardar Patel University
Mathematics
M.Sc. Semester II
Monday, 18 March 2019
10.00 a.m. to 01.00 p.m.
PS02CMTH21 - Real Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $E \subset \mathbb{R}$ and $m^*E = 0$. Then the closure of $\mathbb{R} - E$ is
(a) $\emptyset$ (b) E (c) $\mathbb{R} - E$ (d) $\mathbb{R}$

(2) Let $\{E_n\}$ be a sequence of subsets of $\mathbb{R}$. Which of the following can never happen?
(a) $m^*(\bigcup_n E_n) > \sum_n m^*E_n$ (c) $m^*(\bigcup_n E_n) = \sum_n m^*E_n$
(b) $m^*(\bigcup_n E_n) < \sum_n m^*E_n$ (d) $m^*E_1 = \sum_n m^*E_n$

(3) Let $f : [0,1] \to \mathbb{R}$ be $f = 2\chi_{[0,1]\cap\mathbb{Q}}$. Then the value of $\underline{\int_0^1} f(x)dx$ is
(a) 0 (b) 1 (c) 2 (d) none of these

(4) If $\varphi = \chi_{[0,1]} + 2\chi_{[2,3]}$, then the value of $\int \varphi$ is _____
(a) 0 (b) 1 (c) 2 (d) 3

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 1$ if $x \in \mathbb{R} - \mathbb{Q}$ and $f(x) = e^{-x^2}$ if $x \in \mathbb{Q}$. Then the value of $\int_{\mathbb{R}} f$ is _____
(a) 0 (b) ∞ (c) $\sqrt{\pi}$ (d) $\frac{\sqrt{\pi}}{2}$

(6) If $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n = \chi_{[-n,\infty)}$ and if $x \in \mathbb{R}$, then $\lim_{n\to\infty} f_n(x)$
(a) is 0 (b) is 1 (c) is ∞ (d) is 0 or 1

(7) Let $f : [0,1] \to \mathbb{R}$ be $f(x) = \sqrt{x}$. Which of the following is not true?
(a) f is a Lipschitz function (c) f is of bounded variation
(b) f is absolutely continuous (d) f is uniformly continuous

(8) If $f : [a,b] \to \mathbb{R}$, then which of the following is true?
(a) $f(b) - f(a) \leq T_a^b(f)$ (c) $T_a^b(f) + N_a^b(f) = P_a^b(f)$
(b) $T_a^b(f) \leq f(b) - f(a)$ (d) $T_a^b(f) = T_a^b(|f|)$

Q.2 Attempt any ***Seven.*** [14]

(a) Show that the union of two algebras over X may not be an algebra.
(b) If $E \subset \mathbb{R}$ and $m^*E = 0$, then show that E is measurable.
(c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then show that f is measurable.
(d) If φ is a simple measurable function vanishing outside a set of finite measure and $\varphi \geq 0$ a.e., then show that $\int \varphi \geq 0$.
(e) Evaluate $\lim_{n\to\infty} \int_1^9 \frac{nx}{1+n^2x^2}dx$.

(P.T.O)

(f) If a nonnegative measurable function f is integrable over a measurable set E, then show that f is finite a.e. in E.

(g) If f is integrable over E and if $c < 0$, then show that $\int_E cf = c\int_E f$.

(h) If $f \in BV[a,b]$, then show that f is bounded.

(i) If f is continuous on $[0,1]$, then evaluate $\lim_{n\to\infty} n\int_0^{\frac{1}{n}} f(t)dt$.

Q.3

(a) Define outer measure set of $E \subset \mathbb{R}$. If $E_1, E_2, \ldots, E_n$ are pairwise disjoint measurable subsets of $\mathbb{R}$, then show that $m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^* E_k$. [6]

(b) Show that there exists a nonmeasurable set. [6]

OR

(b) Let E be a measurable set, and let $f : E \to \mathbb{R}$. If f is measurable, then show that both $|f|$ and f^2 are measurable. Is the converse true? Justify. [6]

Q.4

(c) If f and g are nonnegative measurable functions on a measurable set E and if $c \geq 0$, then show that $\int_E cf = c\int_E f$ and $\int_E (f+g) = \int_E f + \int_E g$. [6]

(d) Let E be a measurable set of finite measure, and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real valued function such that $f_n \to f$ pointwise on E. Show that given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in E - A$. [6]

OR

(d) Let f be a bounded measurable function on a measurable set E of finite measure. Show that $\inf_{f\leq\psi} \int_E \psi(x)dx = \sup_{f\geq\varphi} \int_E \varphi(x)dx$, where φ and ψ are measurable simple functions. [6]

Q.5

(e) Let g be integrable over E, and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and $f(x) = \lim_n f_n(x)$ for almost all x in E. Show that $\int_E f = \lim_n \int_E f_n$. [6]

(f) State Monotone Convergence Theorem. Show that the conclusion of the Monotone Convergence Theorem may not hold if the sequence $\{f_n\}$ is decreasing. [6]

OR

(f) Show that convergence in measure may not imply pointwise convergence. [6]

Q.6

(g) If f is absolutely continuous on $[a,b]$ and $f' = 0$ a.e. in $[a,b]$, then show that f is a constant function. [6]

(h) Suppose $f : [a,b] \to \mathbb{R}$ is differentiable and f' is continuous over $[a,b]$. Show that $\int_a^b |f'| = T_a^b(f)$. [6]

OR

(h) If $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then show that f is of bounded variation. Is the converse true? Justify. [6]