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Seat No. _______

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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Tuesday, 26th March, 2019

PS02CMTH04, Functional Analysis - I

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: (1) Figures to the right indicate marks of the respective question.
(2) Here H denotes the Hilbert space over the field K, where K is $\mathbb{R}$ or $\mathbb{C}$, and I denotes identity operator. Assume other usual/standard notations wherever applicable.

Q-1 Write the most appropriate option only for each of the following questions. [08]

1. For $x = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in \ell^2$, $\langle x, x\rangle =$ _______.
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi^2}{36}$ (c) $\frac{\pi}{\sqrt{6}}$ (d) $\frac{\pi^2}{6}$
2. _______ is a Hilbert space.
   (a) ℓ^∞ (b) c_{00} (c) $\mathbb{R}^6$ (d) $C[0,1]$
3. If $x_1, x_2, \ldots, x_n$ are orthogonal, then their Gram matrix is _______ matrix.
   (a) diagonal (b) identity (c) invertible (d) zero
4. If $\{u_n\}$ is an orthonormal basis of an infinite-dimensional Hilbert space, then _____.
   (a) $\|u_n\| \to \sqrt{2}$ (b) $u_n \to 0$ (c) $u_n \xrightarrow{w} 0$ (d) $\{u_n\}$ is Cauchy
5. If S is self-adjoint, $\alpha \in \mathbb{C}$ with real part $\operatorname{Re}\alpha$, then ______ is self-adjoint.
   (a) $i\alpha S$ (b) $-(\operatorname{Re}\alpha)S$ (c) $\bar{\alpha}S$ (d) $i(\operatorname{Re}\alpha)S$
6. If $T \in BL(H)$ be such that T^* is bounded below, then _______.
   (a) T^* is regular (b) T is one-one (c) T^* is onto (d) T is onto
7. Let $T \in BL(\mathbb{R}^7)$ be a projection. Then $\sigma(T) =$_______.
   (a) $\{0,1\}$ (b) $\mathbb{R}$ (c) $\{1,-1\}$ (d) $\emptyset$
8. Let H be a Hilbert space and $T \in BL(H)$. If $\lambda \notin \sigma_a(T)$, then _______.
   (a) $\lambda \notin \sigma_e(T)$ (b) $\lambda \notin \sigma(T)$ (c) $\lambda \notin \overline{W(T)}$ (d) $\lambda \in \sigma_e(T)$

Q-2 Attempt *any seven* of the following. [14]

(a) Show that inner product is jointly continuous.

(b) Let X be an inner product space. Prove that $\|\cdot\|$ induced by the inner product satisfies $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in X$.

(c) Show that $\langle\cdot,\cdot\rangle$ defined by $\langle x, y\rangle = \sum_{i=1}^{n} x(i)\overline{y(i)}$, $\forall\, x, y \in K^n$ is an inner product.

(d) Define best approximation.

(e) Let $P \in BL(H)$ be a projection. Show that $I - P$ is also a projection.

(f) For $T \in BL(H)$, show that $\ker(T) = \ker(T^*T)$.

(g) Let H be a Hilbert space, $T \in BL(H)$ with $T^*T = I$. Show that T is an isometry.

(h) Let H be a Hilbert space. For $T \in BL(H)$ show that $\lambda \in \sigma(T) \Leftrightarrow \bar{\lambda} \in \sigma(T^*)$.

(i) Give an example of a compact operator on a Hilbert space.

(P.T.O.)

Q-3 (a) Let X be an inner product space. Prove that $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$, for all $x,y \in X$ and the equality holds if and only if x and y are linearly dependent. [06]

(b) State and prove Bessel's inequality. [06]

OR

(b) Let H be an infinite dimensional Hilbert space with a countable orthonormal basis. Prove that H is isometrically isomorphic to ℓ^2. [06]

Q-4 (a) State and prove Riesz-representation theorem. [06]

(b) Let X be an inner product space, Y be a subspace of X and $x \in X$. Prove that $y \in Y$ is a best approximation from Y to x if and only if $(x-y) \perp Y$. [06]

OR

(b) Show by an example that completeness of the space is necessary condition in the Projection theorem. [06]

Q-5 (a) Let H be a Hilbert space and $T \in BL(H)$. Prove that there exists a unique $T^* \in BL(H)$ such that $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for every $x,y \in H$. [06]

(b) Let H be a Hilbert space and $T \in BL(H)$ be self-adjoint. Prove that [06]

$$\|T\| = \sup\{|\langle Tx,x\rangle| : x \in H, \|x\| \leq 1\}.$$

OR

(b) Give an example with proper verification of each of the following. [06]

1. A normal operator which is not self-adjoint.
2. A normal operator which is not unitary.

Q-6 (a) Let H be a separable Hilbert space and T be a Hilbert-Schmidt operator on H. Prove that T is compact. [06]

(b) Let $T \in BL(H)$. Prove that $\sigma(T) = \sigma_a(T) \cup \{\bar{\mu} \mid \mu \in \sigma_e(T^*)\}$. [06]

OR

(b) Let H be a Hilbert space and $T \in BL(H)$. Prove that T^* is compact if T is compact. [06]

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