[50/A-5] SEAT No.________

No of printed pages: 2

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M.Sc. (Sem-II), PS02CMTH02, Algebra-I;
Wednesday, 20^{th} March, 2019; 10.00 a.m. to 01.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. Which of the following is a unit in the ring $(\mathbb{Z}, +, \cdot)$?
   (A) 0 (B) -1 (C) 2 (D) 3
2. The ideal generated by $x^2 + 2$ in $\mathbb{R}[x]$ is
   (A) maximal (B) prime but not maximal
   (C) maximal but not prime (D) none of these
3. The polynomial $x^2 + 1$ is reducible over
   (A) $\mathbb{Q}$ (B) $\mathbb{R}$ (C) $\mathbb{C}$ (D) none of these
4. Which is not unit in $J[i]$?
   (A) i (B) $-i$ (C) 1 (D) -2
5. The content of a polynomial $2x^2 + x + 5$ is
   (A) 2 (B) 4 (C) 8 (D) 1
6. $[\mathbb{C} : \mathbb{R}] =$
   (A) 1 (B) 2 (C) 3 (D) ∞
7. The polynomial $x^2 - 2 \in \mathbb{Q}[x]$ is
   (A) solvable by radicals (B) reducible over $\mathbb{Q}$
   (C) not solvable by radicals (D) none of these
8. The number of elements in $G(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{2}))$ is
   (A) 2 (B) 1 (C) 3 (D) ∞

Q.2 Attempt any *seven*: [14]

(a) Show that every field is Euclidean ring.
(b) If R is an Euclidean ring and $a, b, c \in R$ with $a \mid b, b \mid c$ then show that $a \mid c$.
(c) Find units in $J[i]$.
(d) Let $f(x) \in F[x]$ and $a \in F \setminus \{0\}$. If $f(ax)$ is irreducible over F then show that $f(x)$ is irreducible over F.
(e) State Eisenstein criterion.
(f) Prove or disprove : $\mathbb{C}$ is an algebraic extension of $\mathbb{R}$.
(g) Find $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$.
(h) Define radical extension.
(i) Define solvable group.

(1) (P.T.O.)

Q.3

(a) Let R be a Euclidean ring and $a, b \in R \setminus \{0\}$ and b is not unit, then show that $d(a) < d(ab)$. [6]

(b) Show that any nonzero element in Euclidean ring is either a unit or can be written as product of finite number of prime elements. [6]

OR

(b) Let R be an Euclidean ring. Show that the ideal $< a >$ is a maximal ideal in R if and only if a is a prime element in R.

Q.4

(a) Prove that the product of any two primitive polynomials is a primitive polynomial. [6]

(b) Prove that every reducible primitive polynomial over $\mathbb{Q}$ is reducible over $\mathbb{Z}$. [6]

OR

(b) Construct a field of order 121.

Q.5

(a) Let K be an extension field of a field F and $a \in K$ is algebraic over F. Then show that $[F(a) : F] < \infty$. [6]

(b) If K is an extension of F and $a, b \in K$ are algebraic over F. Then show that $a + b$ is algebraic over F. [6]

OR

(b) Find the degree of splitting field of $x^3 - 2$ over $\mathbb{Q}$.

Q.6

(a) Show that K is a normal extension of F, if K is the splitting field of some polynomial over F. [6]

(b) Show that the group S_n, $n \geq 5$ is not solvable. [6]

OR

(b) State the fundamental theorem of Galois theory.