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[52/A8] **SEAT No.________**

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Sardar Patel University

Mathematics

M.Sc. Semester II

Monday, 18 March 2019

10.00 a.m. to 01.00 p.m.

PS02CMTH01 - Real Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $A = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$. Then m^*A is

(a) 0 (b) 1 (c) 2 (d) ∞

(2) For $n \in \mathbb{N}$, let $E_n = [n, n+1]$. Then $m(\bigcup_{n=1}^{\infty} E_n)$ equals ________

(a) 1 (b) 2 (c) 4 (d) ∞

(3) Let $\varphi = 1\chi_{[2,3]} + 3\chi_{[4,5]}$. Then the value of $\int \varphi$ is ____

(a) 1 (b) 2 (c) 3 (d) 4

(4) For $n \in \mathbb{N}$, let $f_n = n\chi_{[0,\frac{1}{n}]}$, and let f be the limit function of $\{f_n\}$. Let $\alpha = \lim_n \int_{[0,1]} f_n$ and $\beta = \int_{[0,1]} f$. Then

(a) $\beta = 1$ (b) $\alpha = \beta$ (c) $\alpha < \beta$ (d) $\alpha > \beta$

(5) Let E have measure 0, and let $f(x) = \infty$ for all $x \in E$. Then $\int_E f$ is

(a) 0 (b) 1 (c) 2 (d) ∞

(6) If f is integrable over E, then the value of $\int_E |f|$ is

(a) $\int_E f^+$ (b) $\int_E f^-$ (c) $\int_E f^+ + \int_E f^-$ (d) $\int_E f^+ - \int_E f^-$

(7) The total variation of $f(x) = \sin^2 x$ on $[0, \frac{\pi}{2}]$ is

(a) 0 (b) $|\sin 1|$ (c) π (d) 1

(8) Which of the following functions is not absolutely continuous on $[0, 1]$?

(a) x^2 (b) e^x (c) $\sin \frac{1}{x}$ (d) $\cos x^2$

Q.2 Attempt any ***Seven.*** [14]

(a) If $E \subset \mathbb{R}$ is a G_δ- set, then show that E^c is an F_σ- set.

(b) If E is a countable set, then show that $m^*E = 0$.

(c) If f_1 and f_2 are measurable on E, then show that $\max\{f_1, f_2\}$ is measurable.

(d) If φ is a measurable simple function vanishing outside a set of finite measure and if $\varphi \geq 0$, then show that $\int \varphi \geq 0$.

(e) Let f be a bounded measurable function defined on a set of finite measure E. If $f \geq 0$ and if F is a measurable subset of E, then show that $\int_F f \leq \int_E f$.

(f) If a nonnegative measurable function f is integrable over E, then show that f is integrable over every measurable subset of E.

(P.T.O)

(g) If f is integrable over E and if $c \in \mathbb{R}$, then show that $\int_E cf = c \int_E f$.
(h) If $f : [a, b] \to \mathbb{R}$ is of bounded variation, then show that f is bounded.
(i) If f is absolutely continuous on $[a, b]$ and $\alpha \in \mathbb{R}$, then show that αf is absolutely continuous on $[a, b]$.

Q.3

(a) Show that the outer measure of an interval is its length. [6]

(b) If f and g are real valued measurable functions on E and if $c \in \mathbb{R}$, then show that both cf and $f + g$ are measurable. [6]

OR

(b) If $\{f_n\}$ is a sequence of measurable function on a measurable set E, then show that $\liminf_n f_n$ and $\limsup_n f_n$ are measurable. [6]

Q.4

(c) Let f be defined and bounded on a measurable set E with mE finite. Suppose that $\inf_{f \le \psi} \int_E \psi(x)dx = \sup_{f \ge \varphi} \int_E \varphi(x)dx$, where φ and ψ are simple functions. Show that f is measurable. [6]

(d) Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure, and let $|f_n| \le M$ on E for all $n \in \mathbb{N}$ for some $M > 0$. If $f(x) = \lim_n f_n(x)$ for each x in E, then show that $\int_E f = \lim_n \int_E f_n$. [6]

OR

(d) If f and g are nonnegative measurable functions on a measurable set E, then show that $\int_E (f + g) = \int_E f + \int_E g$. [6]

Q.5

(e) Let f be a nonnegative function which is integrable over a measurable set E. Show that given $\epsilon > 0$ there is $\delta > 0$ such that $\int_A f < \epsilon$ whenever A is a measurable subset of E with $mA < \delta$. [6]

(f) Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f on E. Then show that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges to f almost everywhere on E. [6]

OR

(f) If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then show that $\int_E f \le \liminf_n \int_E f_n$. [6]

Q.6

(g) Let f be an integrable function on $[a, b]$, and let $F(x) = F(a) + \int_a^x f$ for all $x \in [a, b]$. Show that $F'(x) = f(x)$ for almost all x in $[a, b]$. [6]

(h) Show that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a, b]$. [6]

OR

(h) Let f be integrable on $[a, b]$. Show that $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$ if and only if $f = 0$ a.e. in $[a, b]$. [6]

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