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[37/A-23]

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Tuesday, 17th April, 2018

PS02CMTH24, Functional Analysis-I

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: Figures to the right indicate full marks of the respective questions.
Assume standard notations wherever applicable. H is a Hilbert space.

Q-1 Write the question number and appropriate **option number only** for each question. [8]

(a) A/An ____ set is ____ but the converse is not true.
(i) orthonormal, orthogonal (iii) orthogonal, orthonormal
(ii) linearly independent, orthogonal (iv) all of these

(b) For $x \in \mathbb{R}^2$, if $\langle x, (1,1)\rangle = 7$, $\langle x, (-1,1)\rangle = 1$, then $\|x\| =$ ____
(i) 5 (ii) 6 (iii) 7 (iv) 8

(c) Best approximation from $E = \mathbb{R} \times \{0\}$ to $(3,3)$ is
(i) 0 (ii) (0,0) (iii) (3,0) (iv) (3,3)

(d) For $\{u_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), u_2 = (-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}})\}$ equality holds in Bessel's inequality for $x =$ ____
(i) $(1,2,3)$ (ii) $(2,4,2)$ (iii) $(1,2,1)$ (iv) $(0,0,1)$

(e) For $T \in BL(H)$, ____ is selfadjoint.
(i) $T - T^*$ (ii) $T^* - T$ (iii) TT^* (iv) $\frac{T^2+T}{2}$

(f) For $T \in BL(H)$ if $\|Tx\| = \|T^*x\|$ for all $x \in H$, then T is ____.
(i) normal (ii) selfadjoint (iii) unitary (iv) compact

(g) For a/an ____ operator T on a ℓ^2, $0 \in \sigma(T)$.
(i) normal (ii) selfadjoint (iii) unitary (iv) compact

(h) For a ____ operator T on a ℓ^2, amd $x \in H$, $T^2x = 0 \Rightarrow Tx = 0$.
(i) normal (ii) compact (iii) Hilbert-Schmidt (iv) all of these

Q-2 Attempt *Any Seven* of the following: [14]

(a) If $x_n \to x$ and $y_n \to y$ in H, then show that $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

(b) Given an example to show that Gram-Schmidt orthonormalization of $\{x_1, x_2\}$ and $\{x_2, x_1\}$ may give rise to different orthonormal sets.

(c) Give an example to show that best approximation may not exist.

(d) Find the representer of the linear functional $f : \mathbb{C}^3 \to \mathbb{C}$ defined by $f(x, y, z) = (2+3i)x + 4y - iz$, $((x, y, z) \in \mathbb{C}^3)$.

(e) Prove the continuity of $T : \ell^2 \to \ell^2$ defined by $T(x(1), x(2), x(3), \ldots) = (x(1), \frac{x(1)}{2}, \frac{x(1)}{3}, \ldots)$.

(f) Define the adjoint of an operator. Find adjoint of $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T(z_1, z_2, z_3) = (2z_2 + iz_3, z_1 + z_2, z_3)$, $(z_1, z_2, z_3) \in \mathbb{C}^3$. Justify your answer.

(g) If $T \in BL(H)$ is bounded below, then show that $R(T)$ is closed in H.

(h) For a normal operator $T \in BL(H)$, show that $\|T^2\| = \|T\|^2$

(i) If range $R(T)$ of $T \in BL(H)$ is finite dimensional, then show that T is compact.

(P. T. O.)

Q-3 (j) State and prove Riesz-Fischer's theorem [6]

(k) Show that an inner product space is a uniformly convex normed linear space with the induced norm. [6]

OR

(k) Show that if an infinite dimensional Hilbert space H has a countable orthonormal basis, then it is isometrically isomorphic to ℓ^2. [6]

Q-4 (l) Let $E \subset H$. Show that $E^{\perp}$ is a closed subspace of H. [6]

(m) Let X be an inner product space and $x_1, x_2, \ldots, x_n \in X$ be linearly independent. Show that the Gram matrix of $x_1, x_2, \ldots, x_n$ is regular. [6]

OR

(m) Let X be an inner product space, Y be a subspace of X and $x \in X$. If $y \in Y$ is a best approximation from Y to x, then show that $(x - y) \perp Y$. [6]

Q-5 (n) Define and prove existence of adjoint of $T \in BL(H)$. [6]

(o) Let $S, T \in BL(H)$ be normal. If S commutes with T^*, then show that $S + T$ is normal. If S, T are unitary, then show that ST is also unitary. [6]

OR

(o) Show that $T \in BL(H)$ is an isometry if and only if $T^*T = I$. [6]

Q-6 (p) For $T \in BL(H)$, show that $\sigma(T) = \sigma_a(T) \cup \{\bar{\mu} : \mu \in \sigma_e(T^*)\}$. [6]

(q) For a normal $T \in BL(H)$, show that the eigenvectors corresponding to distinct eigenvalues of T are orthogonal. Also, show that the orthogonality may not hold if T is not normal. Verify your claim. [6]

OR

(q) For a selfadjoint $T \in BL(H)$, show that $m_T \in \sigma_a(T)$. [6]

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