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SEAT No.________

[38/A-26]

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M.Sc. (Sem-II), PS02CMTH22, Algebra-I;

Wednesday, 11^{th} April, 2018; 10.00 a.m. to 01.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. Pick out the maximal ideal of $(\mathbb{Z}, +, \cdot)$ from the following.
   (A) $< 121 >$ (B) $< 4 >$ (C) $< 1 >$ (D) $< 11 >$
2. Which is false from the following ?
   (A) Every integral domain can be imbedded in a division ring.
   (B) A finite integral domain is a field.
   (C) Every field is principal ideal ring.
   (D) none of these
3. The number $\sqrt{5+\sqrt{5}}$ is algebraic over $\mathbb{Q}(\sqrt{5})$ of degree
   (A) 1 (B) 5 (C) 2 (D) 4
4. Which is true from the following ?
   (A) If a is transcendental and b is algebraic, then $a+b$ is algebraic.
   (B) If a is transcendental and $b \neq 0$ is algebraic, then ab^{-1} is algebraic.
   (C) If ab is transcendental, then at least one of a or b is transcendental.
   (D) none of these
5. What is $o(G(K, \mathbb{Z}_3))$, where K is the splitting field of x^2+2 over $\mathbb{Z}_3$?
   (A) 2 (B) 1 (C) 3 (D) none of these
6. The degree of the splitting field of x^4+x^2+1 over $\mathbb{Q}$ is
   (A) 1 (B) 2 (C) 3 (D) 4
7. The Galois group of $x^2-7 \in \mathbb{Q}[x]$ is
   (A) cyclic (B) abelian but not cyclic
   (C) non abelian simple (D) none of these
8. Which one from the following is not solvable ?
   (A) S_3 (B) $(\mathbb{Z}_5, +)$ (C) S_4 (D) none of these

Q.2 Attempt any ***seven***: [14]

(a) Define Euclidean ring.
(b) Find all units of the ring of Gaussian integers, $J[i] = \{a+bi : a, b \in \mathbb{Z}\}$.
(c) Let $F \subset K$ and $a \in K$ be a root of $p(x) \in F[x]$. Show that $(x-a) \mid p(x)$ in $K[x]$.
(d) Is $\sin 2^0$ algebraic over $\mathbb{Q}$? Justify.
(e) Show that any non constant irreducible polynomial over a field of characteristic zero can not have multiple root.
(f) Show that $K_G = \{a \in K : \sigma(a) = a, \forall \sigma \in G\}$ is a subfield of K, where $G = Aut(K)$.
(g) Define normal extension and give one example of it.
(h) Is $\mathbb{Q}(2^{\frac{1}{2}}, 3^{\frac{1}{3}}, 5^{\frac{1}{5}})$ a radical extension of $\mathbb{Q}$? Justify.
(i) Define solvable group.

(P.T.O.)

Q.3

(a) State and prove Unique Factorization Theorem. [6]

(b) Show that any two non zero elements in Euclidean ring have the greatest common divisor and the least common multiple in the Euclidean ring. [6]

OR

(b) State and and prove the necessary and sufficient condition that the principle ideal in Euclidean ring to be maximal.

Q.4

(a) If $F \subset K$ and $a \in K$ is algebraic over F, then show that $[F(a) : F]$ is finite. [6]

(b) Let $p(x)$ be nonconstant irreducible polynomial of degree n over F. Then show that there exists an extension E of F having $[E : F] = n$ such that $p(x)$ has a root in E. [6]

OR

(b) Construct a field containing exactly 27 elements. State results which you use.

Q.5

(a) Let $\sigma_i \in Aut(K)$ be distinct for $i = 1, 2, ..., n$. Can we find $a_i \in K$ $(i = 1, 2, ..., n)$ not all zero satisfying $\sum_{i=1}^{n} a_i \sigma_i(u) = 0$, $\forall u \in K$? Justify. [6]

(b) If F and F' are isomorphic fields, then show that any splitting fields E and E' of polynomials $f(x) \in F[x]$ and $f'(t) \in F'[t]$ respectively, are isomorphic by an isomorphism, ϕ with $\phi(\alpha) = \alpha'$, $\forall \alpha \in F$. [6]

OR

(b) Find the degree of the splitting field of $x^p - 1$ over $\mathbb{Q}$, where $p > 2$ is a prime number.

Q.6

(a) Prove that S_n is not solvable, where $n \geq 5$. [6]

(b) Let K be a normal extension of F, H be a subgroup of $G(K, F)$ and K_H be a fixed field of H. Then show that $[K : K_H] = o(H)$. [6]

OR

(b) State and prove Able's theorem. State results which you use.

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