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SEAT No.________

[548 A-22]

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Sardar Patel University
Mathematics
M.Sc. Semester II
Monday, 9 April 2018
10.00 a.m. to 01.00 p.m.
PS02CMTH21 - Real Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $E = \mathbb{Q} \cap (-1, 1]$, and let $A \subset \mathbb{R}$. Which of the following is true?

(a) $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ (c) $m^*(A) < m^*(A \cap E) + m^*(A \cap E^c)$
(b) $m^*(A) > m^*(A \cap E) + m^*(A \cap E^c)$ (d) $m^*(E) \geq m^*(A \cap E) + m^*(A \cap E^c)$

(2) Let $f, g : \mathbb{R} \to \mathbb{R}$, let f be differentiable and $g = f$ a.e. in $\mathbb{R}$. Then

(a) g is continuous (c) g is measurable
(b) g is differentiable (d) g is a constant map

(3) Let f be a bounded measurable function vanishing outside a set of finite measure. Let F be a measurable subset of a measurable set E. Which of the following is true?

(a) $\int_E f = \int_F f$ (b) $\int_E f \leq \int_F f$ (c) $\int_E f \geq \int_F f$ (d) none of these

(4) Let $f : [a, b] \to \mathbb{R}$ be bounded and A be the set of continuity of f. Then f is Riemann integrable if and only if _____

(a) A is a finite set (c) A has measure 0
(b) A is a countable set (d) $mA = b - a$

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be an even function, $f(n) = 0$ for all $n \in \mathbb{Z}$, and let $f(x) = \frac{1}{n^2}$ if $n - 1 < x < n$ for $n \in \mathbb{N}$. Then the value of $\int_{\mathbb{R}} f$ is _____

(a) 0 (b) 1 (c) ∞ (d) none of these

(6) If f is integrable function over a measurable set E, then _____

(a) $f = 0$ a.e. (b) f is finite a.e. (c) $f \geq 0$ a.e. (d) $\int_E f \geq 0$

(7) The total variation of $f(x) = x^3$ over $[0, 2]$ is _____

(a) 1 (b) 2 (c) 4 (d) 8

(8) If $f : [a, b] \to \mathbb{R}$, then _____

(a) $T_a^b = P_a^b + N_a^b$ (b) $T_a^b = P_a^b - N_a^b$ (c) $T_a^b = N_a^b - P_a^b$ (d) $T_a^b = -N_a^b - P_a^b$

Q.2 Attempt any ***Seven***. [14]

(a) Show that $[1, 2]$ is a G_δ- set.
(b) If f^2 is a measurable function, then show that f may not be measurable.
(c) If $f_1, f_2, \ldots, f_n$ are measurable functions on E, then show that $\min\{f_1, f_2, \ldots, f_n\}$ is a measurable function.
(d) If φ is a nonnegative measurable simple function vanishing outside a set of finite measure, then show that $\int_E \varphi \geq 0$ for every measurable set E.

(P.T.O.)

(e) Let $f : [0,1] \to \mathbb{R}$ be a continuous map. Evaluate $\lim_{n\to\infty} \int_0^1 f(x)x^n dx$.

(f) Let f be a nonnegative measurable function on a measurable set E. If $\int_E f = 0$, then show that $f = 0$ almost everywhere.

(g) If f is integrable over E, then show that $\left|\int_E f\right| \leq \int_E |f|$.

(h) Let $f \in BV[a,b]$ and $f(x) \neq 0$ for any $x \in [a,b]$. Is $\frac{1}{f} \in BV[a,b]$? Why?

(i) Is $\sin\frac{1}{x}$ absolutely continuous on $[1,2]$? Justify.

Q.3

(a) Define outer measure set of $E \subset \mathbb{R}$. When is a set $E \subset \mathbb{R}$ called measurable? If $\{E_n\}$ is a sequence of measurable sets, then show that $\bigcup_n E_n$ is measurable. [6]

(b) Show that (a, ∞) is measurable for all $a \in \mathbb{R}$. [6]

OR

(b) Let E be a measurable set. When is a function $f : E \to [-\infty, \infty]$ called measurable. Let f be an extended real valued function on a measurable set E? Show that f is measurable if and only if $f^{-1}(U)$ is measurable for every open subset U of $[-\infty, \infty]$. [6]

Q.4

(c) If $f : [a,b] \to \mathbb{R}$ is a bounded function, then show that $\underline{\int_a^b} f(x)dx = \sup_{\phi \leq f} \int_a^b \phi(x)dx$, where ϕ is a step function over $[a,b]$. [6]

(d) If $\{f_n\}$ is a sequence of measurable functions that converge to a real valued function f a.e. on a measurable set E of finite measure, then show that given $\eta > 0$, there is a measurable subset A of E with $mA < \eta$ such that $\{f_n\}$ converges to f uniformly on $E - A$. Also, show that the conclusion may not hold if mE is infinite. [6]

OR

(d) Let f be a real valued measurable function on a measurable set E of finite measure. Then show that for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F and $m(E - F) < \epsilon$. [6]

Q.5

(e) If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then show that $\int_E f \leq \liminf_n \int_E f_n$. Also, give and example of a sequence $\{f_n\}$ and a functions f for which we get the strict inequality. [6]

(f) Let f be integrable over a measurable set E, and let $\epsilon > 0$. Show that there is $\delta > 0$ such that $\left|\int_A f\right| < \epsilon$ for every measurable subset A of E with $mA < \delta$. [6]

OR

(f) Let f_n, f be measurable functions on E. When do we say that $\{f_n\}$ converges to f in measure? Suppose that a sequence $\{f_n\}$ converges to f pointwise on E. Is it true that $f_n \to f$ in measure on E? Justify. [6]

Q.6

(g) If f is absolutely continuous on $[a,b]$ and $f' = 0$ a.e. in $[a,b]$, then show that f is a constant function. [6]

(h) Let $f, g : [0,1] \to \mathbb{R}$ be defined as $f(0) = g(0) = 0$, $f(x) = x\sin\frac{1}{x}$, $0 < x \leq 1$ and $g(x) = x^2 \sin\frac{1}{x^2}$, $0 < x \leq 1$. Are f and g of bounded variation? Justify. [6]

OR

(h) Let f be an integrable function over $[a,b]$, and let $F(x) = F(a) + \int_a^x f$ for all $x \in [a,b]$. Show that $F'(x) = f(x)$ for almost all x in $[a,b]$. [6]