Sc

Seat No.: _____ No. of printed pages: 2

[36/A-22]

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Tuesday, 17^{th} April, 2018

PS02CMTH04, Functional Analysis-I

Time: 10:00 a.m. to 01:00 p.m. Maximum marks: 70

Note: Figures to the right indicate full marks of the respective questions.
Assume standard notations wherever applicable. H is a Hilbert space.

Q-1 Write the question number and appropriate **option number only** for each question. [8]

(a) ℓ^p is an inner product space if $p =$_____

(i) 2 (ii) 2 or ∞ (iii) 1 (iv) all of these

(b) For a real number p, $\|x\|_p = (\sum_{i=1}^n |x(i)|^p)^{\frac{1}{p}}$ is a norm on $\mathbb{C}^n$ if and only if p _____.

(i) > 0 (ii) > 1 (iii) ≥ 1 (iv) none of these

(c) Best approximation from $E = \mathbb{R} \times \mathbb{N}$ to $(3, 3)$ is

(i) 0 (ii) (0,0) (iii) (3,0) (iv) $(3, 3)$

(d) For $\{u_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), u_2 = (-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}})\}$ equality holds in Bessel's inequality for $x =$ _____

(i) $(1, 2, 3)$ (ii) $(2, 4, 2)$ (iii) $(1, 2, 1)$ (iv) $(0, 0, 1)$

(e) For $T \in BL(H)$, _____ is selfadjoint.

(i) $T - T^*$ (ii) $T^* - T$ (iii) TT^* (iv) $\frac{T^2+T}{2}$

(f) A _____ operator is invertible.

(i) normal (ii) selfadjoint (iii) unitary (iv) compact

(g) For a/an _____ operator T on a ℓ^2, $0 \in \sigma(T)$.

(i) normal (ii) selfadjoint (iii) unitary (iv) compact

(h) If numerical range of _____ operator T is $\{0\}$, then $T = 0$.

(i) normal (ii) compact (iii) Hilbert-Schmidt (iv) positive

Q-2 Attempt *Any Seven* of the following: [14]

(a) Define and give an example of a *normed linear space.*

(b) For $x, y \in H$, prove that $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

(c) Find value of $a \in \mathbb{R}$ such that $\{(a, 1), (2, 3)\}$ is orthogonal.

(d) Define and give an example of a *projection.* Verify that it is a projection.

(e) Define the *right shift operator* on ℓ^2 and find its kernel.

(f) Define the adjoint of an operator. Find adjoint of the operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T(z_1, z_2, z_3) = (2z_2 + iz_3, z_1 + z_2, z_3)$, $(z_1, z_2, z_3) \in \mathbb{C}^3$. Justify your answer.

(g) For operators $T, S \in BL(H)$, prove that $(ST)^* = T^*S^*$

(h) Define *spectrum and eigenspectrum* of an operator.

(i) Define and give an example of a *compact operator.* Justify your claim.

[P. T. O.]

Q-3 (j) State and prove the Schwarz inequality for inner product space. [6]

(k) Show that an inner product space is a uniformly convex normed linear space with the induced norm. [6]

OR

(k) Let X be an inner product space, E be an orthonormal subset of X and $x \in X$. Show that the set $E_x = \{u \in E : \langle x, u \rangle \neq 0\}$ is countable. [6]

Q-4 (l) Let $E \subset H$. Show that $E^{\perp}$ is a closed subspace of H. [6]

(m) Let X be an inner product space and $x_1, x_2, \ldots, x_n \in X$ be linearly independent. Show that the Gram matrix of $x_1, x_2, \ldots, x_n$ is regular. [6]

OR

(m) Let X be an inner product space. If $E \subset X$ is convex and $x \in X$, then show that there exists at most one best approximation from E to x. [6]

Q-5 (n) For $T \in BL(H)$, show that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$. [6]

(o) (i) For $T \in BL(H)$, show that $\ker(T)^{\perp} = \overline{R(T^*)}$. Hence deduce that T is one-one if and only if $R(T^*)$ is dense in H. [6]

(ii) Show that $T \in BL(H)$ is an isometry if and only if $T^*T = I$.

OR

(o) Given $S \in BL(H)$, show that there are unique selfadjoint operators $A, B \in BL(H)$ such that $S = A + iB$. [6]

Q-6 (p) For $T \in BL(H)$, show that $\sigma_e(T) \subset \sigma_a(T)$. Also give an example to show that inclusion may be proper. Justify your claim. [6]

(q) (i) If $T \in BL(H)$ is normal, then show that $\sigma(T) = \sigma_a(T)$. [6]

(ii) For $T \in BL(H)$, show that $\sigma_a(T) \subset \overline{W(T)}$.

OR

(q) Show that a compact linear transformation $T : H \to H$ is bounded. [6]

---- x ----