SC

SEAT No.________

No of printed pages: 2

[538 A-21]

Sardar Patel University
Mathematics
M.Sc. Semester II
Monday, 09 April 2018
10.00 a.m. to 01.00 p.m.
PS02CMTH01 - Real Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $A = (0, 1)$ and $B = [0, 1] - \mathbb{Q}$. Which of the following is true?

(a) $m^*A < m^*B$ (b) $m^*A > m^*B$ (c) $m^*A = m^*B$ (d) none of these

(2) For $n \in \mathbb{N}$, let $E_n = [-\frac{1}{n}, 1] \cup [2, 2 + \frac{1}{n^2}]$. Then $m(\bigcap_{n=1}^{\infty} E_n)$ equals

(a) 1 (b) 2 (c) 4 (d) ∞

(3) Let $f : [0, 1] \to \mathbb{R}$ be defined as $f(x) = 1$ if $x \in \mathbb{R} - \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{Q}$. The the value of the Lebesgue integral $\int_0^1 f(x)dx$ is ____

(a) 0 (b) 1 (c) 2 (d) ∞

(4) The value of $\lim\limits_{n\to\infty} \int\limits_{[0,1]} e^x x^n dx$ is ____

(a) 0 (b) 1 (c) ∞ (d) none of these

(5) Which of the following is true for a function f?

(a) $f = f^+ + f^-$ (b) $|f| = f^+ - f^-$ (c) $f = f^+ - f^-$ (d) $|f| = -f^+ - f^-$

(6) Let f be measurable on E. Which of the following implies that f is integrable over E?

(a) $|f|$ is integrable
(b) f^2 is integrable
(c) f^+ is integrable
(d) f^+ or f^- is integrable

(7) The total variation of $f(x) = \sin x$ on $[0, \frac{\pi}{2}]$ is

(a) 0 (b) $|\sin 1|$ (c) π (d) 1

(8) If $f : [a, b] \to \mathbb{R}$, then which of the following is true?

(a) $T_a^b = P_a^b - N_a^b$ (b) $T_a^b = P_a^b + N_a^b$ (c) $T_a^b = -P_a^b + N_a^b$ (d) $T_a^b = -P_a^b - N_a^b$

Q.2 Attempt any *Seven*. [14]

(a) Let $E \subset \mathbb{R}$ and $m^*E = 0$. Show that E is measurable.
(b) Show that $[1, 2]$ is a G_δ- set.
(c) If $|f|$ is measurable over $\mathbb{R}$, then show that f need not be measurable.
(d) If f is a nonnegative measurable function over a measurable set E and if $\int_E f = 0$, then show that $f = 0$ a.e. on E.
(e) If f is a bounded measurable function defined on a set of finite measure E, then show that $\left|\int_E f\right| \leq \int_E |f|$.

(P.T.O.)

(f) Let $\{f_n\}$ be a sequence of nonnegative measurable functions defined on a measurable set E and $f_n \to f$ on E. If $f_n \leq f$ for each n, then show that $\int_E f = \lim_n \int_E f_n$.
(g) State Lebesgue's dominated convergence theorem.
(h) If $f : [a, b] \to \mathbb{R}$ is decreasing, then show that f is of bounded variation.
(i) Show that every absolutely continuous function is continuous.

Q.3

(a) Define outer measure of $E \subset \mathbb{R}$. If $E_1, E_2, \ldots, E_n$ are measurable, then show that $\bigcup_{k=1}^{n} E_k$ is measurable. [6]

(b) Let E be a measurable set. When is a function $f : E \to [-\infty, \infty]$ called measurable? Show that a function $f : E \to [-\infty, \infty]$ is measurable if and only if the set $\{x \in E : f(x) < r\}$ is measurable for every $r \in \mathbb{Q}$. [6]

OR

(b) Prove that $[0, 1]$ contains a nonmeasurable set. [6]

Q.4

(c) Let f be defined and bounded on a measurable set E with mE finite. Suppose that $\inf_{f\leq\psi} \int_E \psi(x)dx = \sup_{f\geq\varphi} \int_E \varphi(x)dx$, where φ and ψ are simple functions. Show that f is measurable. [6]

(d) If φ and ψ are measurable simple functions vanishing outside a set of finite measure, then prove that $\int(\varphi + \psi) = \int \varphi + \int \psi$. [6]

OR

(d) If $\{f_n\}$ is a sequence of measurable functions that converge to a real valued function f a.e. on a measurable set E of finite measure, then show that given $\epsilon > 0$, there is a measurable subset $A \subset E$ with $mA < \epsilon$ such that $\{f_n\}$ converges to f uniformly on $E - A$. [6]

Q.5

(e) Let $\{u_n\}$ a sequence of nonnegative measurable functions defined on a measurable set E. Prove that $\int_E (\sum_{n=1}^{\infty} u_n) = \sum_{n=1}^{\infty} \int_E u_n$. State the results you use. [6]

(f) If f and g are integrable over E, then prove that $f + g$ is integrable over E and $\int_E (f + g) = \int_E f + \int_E g$. [6]

OR

(f) When do we say that a sequence $\{f_n\}$ converges in measure? Let $\{f_n\}$ be a sequence of measurable functions that converge in measure to f. Show that there is a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere. [6]

Q.6

(g) Let f be an integrable function on $[a, b]$, and let $F(x) = F(a) + \int_a^x f$ for all $x \in [a, b]$. Show that $F'(x) = f(x)$ for almost all x in $[a, b]$. [6]

(h) If f is absolutely continuous on $[a, b]$, then show that f is of bounded variation on $[a, b]$. Is the converse true? Justify. [6]

OR

(h) If f is integrable on $[a, b]$ and $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$, then show that $f = 0$ a.e. in $[a, b]$. [6]

——X——