SEAT No.________

[S3/A-18]

No of printed pages: 2

Sardar Patel University
Mathematics
M.Sc. Semester II
Monday, 10 April 2017
10.00 a.m. to 1.00 p.m.
PS02CMTH01 - Real Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $E = \mathbb{Q} \cap (-1, 1]$. Then $m(E) =$____

(a) 0 (b) 1 (c) 2 (d) none of these

(2) Let $\{x_n : n \in \mathbb{N}\} = \mathbb{Q}$ and $A = \bigcup_n \left(x_n - \frac{1}{n^2}, x_n + \frac{1}{n^2}\right)$. Then ____

(a) $A = \mathbb{R}$ (b) $m(A) = \infty$ (c) $m(A) < \infty$ (d) none of these

(3) Let $E \subset \mathbb{R}$ and $m(E) = 0$. Let $f(x) = 2$ for all $x \in E$. Then $\int_E f =$____

(a) $2m(E)$ (b) 2 (c) $2 + m(E)$ (d) ∞

(4) Let $f : [a, b] \to \mathbb{R}$ be bounded and A be the set of discontinuity of f. Then f is Riemann integrable if and only if ____

(a) A is a finite set
(b) A is a countable set
(c) A has measure 0
(d) $m(A) > 0$

(5) If f is integrable over E, then $\int_E f =$ ____

(a) $\int_E f^+ - \int_E f^-$ (b) $\int_E f^+ + \int_E f^-$ (c) $\int_E f^- - \int_E f^+$ (d) none of these

(6) Let f and g be integrable functions on E such that $\int_E f = \int_E g$. Then ____

(a) $f = g$ a.e. (b) $f \leq g$ (c) $g \leq f$ (d) none of these

(7) The total variation of $f(x) = x^2$ over $[0, 2]$ is ____

(a) 1 (b) 2 (c) 4 (d) 8

(8) Let f be absolutely continuous on $[a, b]$. Which of the following is not true?

(a) f is continuous.
(b) f is of bounded variation.
(c) f is differentiable a.e.
(d) f is differentiable.

Q.2 Attempt any *Seven*. [14]

(a) Show that every countable subset of $\mathbb{R}$ is measurable.
(b) Give an example of a non-measurable function.
(c) If f is measurable, then show that $f^{-1}(a, b)$ is a measurable set.
(d) Let $f : [0, 1] \to \mathbb{R}$ be $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \in \mathbb{R} - \mathbb{Q}$. Is f Riemann integrable? Why?
(e) Let f be a nonnegative measurable function on a measurable set E. If $\int_E f = 0$, then show that $f = 0$ almost everywhere.
(f) State Monotone Convergence Theorem.

(g) Give an example of a sequence $\{f_n\}$ of measurable functions and a measurable function f on a measurable set E such that $f_n \to f$ but $\{\int_E f_n\}$ does not converge to $\int_E f$.

(h) If f and g are absolutely continuous on $[a, b]$, then show that fg is absolutely continuous.

(i) Suppose that $f \leq g$ a.e. on E. Then show that $\int_E f \leq \int_E g$.

Q.3

(a) Let $\{E_n\}$ be a decreasing sequence of measurable subsets of $\mathbb{R}$ and $m(E_1) < \infty$. Show that $m(\bigcap_n E_n) = \lim_n m(E_n)$. Also, show that it is not true if $m(E_k)$ is not finite for some k. [6]

(b) Let $\{f_n\}$ be a sequence of measurable functions. Show that $\liminf_n f_n$ and $\limsup_n f_n$ are measurable. Deduce that $\lim_n f_n$ is measurable if the sequence $\{f_n\}$ is convergent. [6]

OR

(b) Show that the collection $\mathscr{M}$ of measurable subsets of $\mathbb{R}$ is a σ-algebra. [6]

Q.4

(c) Let E be a measurable set of finite measure, and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real valued function such that $f_n(x) \to f(x)$ for all $x \in E$. If $\epsilon > 0$ and $\delta > 0$, then show that there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in E - A$ and $n > N$. [6]

(d) Define Lebesgue integral of a bounded measurable function f on a measurable set E of finite measure. If f, g are measurable bounded measurable functions on a measurable set E of finite measure and $a, b \in \mathbb{R}$, then show that $\int_E (af + bg) = a\int_E f + b\int_E g$. [6]

OR

(d) State and prove Bounded Convergence Theorem. [6]

Q.5

(e) Let f be a nonnegative measurable function on a measurable set E and $\{E_n\}$ be a sequence of pairwise disjoint measurable sets such that $E = \bigcup_n E_n$. Then prove that $\int_E f = \sum_n \int_{E_n} f$. State the results you use. [6]

(f) If f is integrable over E, then show that $|f|$ is integrable over E and $|\int_E f| \leq \int_E |f|$. Does the integrability of $|f|$ imply the integrability of f? Why? [6]

OR

(f) Let f_n, f be measurable functions on E. When do we say that $\{f_n\}$ converges to f in measure? If $\{f_n\}$ converges to f in measure, then show that $\{f_n\}$ has a subsequence which converges to f almost everywhere. [6]

Q.6

(g) When is a function $f : [a, b] \to \mathbb{R}$ called a function of bounded variation? Show that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a, b]$. [6]

(h) If f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$, then show that $F'(x) = f(x)$ for almost all x in $[a, b]$. Also, show that F is of bounded variation on $[a, b]$. [6]

OR

(h) Show that every absolutely continuous function is the indefinite integral of its derivative. [6]

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Seat No.________

[38/A-15]

No of printed pages: 2

Sardar Patel University

Mathematics
M.Sc. Semester II
Saturday, 15 April 2017
10.00 a.m. to 1.00 p.m.
PS02CMTH03 Differential Geometry

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Then $\overline{\gamma}(t) = (a\cos t, a\sin t, bt)$ is planar iff

(a) $a = 0$ or $b = 0$ (b) $a = 0$ (c) $b = 0$ (d) $a \neq 0$ and $b \neq 0$

(2) How many vertices does a circle have?

(a) 0 (b) 1 (c) 4 (d) infinitely many

(3) Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a local diffeomorphism and $p \in \mathcal{S}_1$. Which of the following is not true?

(a) D_pf is linear (b) D_pf is onto (c) D_pf is isometry (d) D_pf is one one

(4) Minimum number of patches required to cover circular cylinder $x^2 + y^2 = 1$ is

(a) 1 (b) 2 (c) 4 (d) 6

(5) Which of the following is not an oriented surface?

(a) Mobius strip (b) torus (c) plane (d) sphere

(6) The mean curvature of $\sigma(u, v) = (2u, 3v, 2u + 3v)$ at the point $(2, 3, 5)$ is

(a) 2 (b) 3 (c) 5 (d) none of these

(7) Let $\overline{\gamma}$ be a curve on a surface $\mathcal{S}$. Then $\overline{\gamma}$ is a geodesic iff

(a) $\kappa_g = 1$ (b) $\kappa_n = 0$ (c) $\kappa_g^2 + \kappa_n^2 = \kappa^2$ (d) $|\kappa_g| = 0$

(8) Let $\overline{\gamma}$ be a curve on a surface. Which of below is a tangent vector field along $\overline{\gamma}$?

(a) $\dot{\overline{\gamma}}$ (b) $\ddot{\overline{\gamma}}$ (c) $\dot{\overline{\gamma}} + \ddot{\overline{\gamma}}$ (d) $\dot{\overline{\gamma}} \times \ddot{\overline{\gamma}}$

Q.2 Attempt any *Seven*. [14]

(a) Compute the signed curvature of $\overline{\gamma}(t) = (\sin t, \cos t)$.

(b) Show that tangent vectors to $\overline{\gamma}(t) = (\cos t, \sin t, t)$ make a constant angle with the z-axis.

(c) Show that a reparametrization of a regular curve is regular.

(d) Show that $\{(x, y, z) : x^2 + y^2 = z^2\}$ is not a surface.

(e) Compute the surface area of $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4,\ x > 0\}$.

(f) Let $\overline{\gamma}$ be a unit-speed curve on an oriented surface. Define geodesic curvature and normal curvature of $\overline{\gamma}$.

(g) Let σ be a surface patch of an oriented surface with the unit normal $\overline{\mathbf{N}}$. Then show that $\overline{\mathbf{N}}_u\sigma_u = -L$, $\overline{\mathbf{N}}_u\sigma_v = -M = \overline{\mathbf{N}}_v\sigma_u$ and $\overline{\mathbf{N}}_v\sigma_v = -N$.

(h) Define Christoffel's symbols of second kind for a regular surface patch $\sigma : U \to \mathbb{R}^3$.
(i) State *Gauss' Theorema Egregium.*

Q.3

(a) Let $\overline{\gamma} : \mathbb{R} \to \mathbb{R}^2$ be a simple closed curve. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a direct isometry. Show that $\overline{\gamma}$ and $M \circ \overline{\gamma}$ have the same length and the same area of interior. [6]

(b) Let $\overline{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature, and let $\overline{\mathbf{t}}$ be the unit tangent of $\overline{\gamma}$. Is the curve $\overline{\alpha} = \overline{\mathbf{t}}$ regular? Find the curvature and torsion of $\overline{\alpha}$. [6]

OR

(b) Show that [6]

$$\int_0^{2\pi} \sqrt{81 \sin^2 t + 16 \cos^2 t}\, dt > 12\pi.$$

State the result you use.

Q.4

(c) Define *smooth surface.* Show that open subset of a smooth surface is a smooth surface. [6]

(d) Explain *stereographic projection.* Hence give an example of a conformal map which is not a local isometry. [6]

OR

(d) Define the *first fundamental form* at a point p on a smooth surface $\mathcal{S}$. Let $\sigma : U \to \mathbb{R}^3$ be a regular patch, and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry. Show that σ and $T \circ \sigma$ have the same first fundamental forms. [6]

Q.5

(e) Let $\overline{\gamma} : (a, b) \to \mathbb{R}^3$ be a unit-speed curve with nowhere vanishing curvature, and let $\overline{\mathbf{t}}$ be the unit tangent of $\overline{\gamma}$. Let $\sigma : (a, b) \times (0, \infty) \to \mathbb{R}^3$ be $\sigma(u, v) = \overline{\gamma}(u) + v\overline{\mathbf{t}}(u)$. Compute the Gaussian curvature and mean curvature of σ. Also, show that the mean curvature of σ is zero if and only if $\overline{\gamma}$ is planar. [6]

(f) Define *principal curvatures* and *principal vectors* at a point on an oriented surface. Compute the principal curvatures of $\sigma(u, v) = ((b+a \cos v) \cos u, (b+a \cos v) \sin u, a \sin v)$, where $b > a > 0$. [6]

OR

(f) When is a point on a surface called *umbilic*? Show that a point p on a surface $\mathcal{S}$ is umbilic if and only if W_p is a scalar multiple of identity. Also, show that every point of $\sigma(u, v) = (u, v, u + v)$ is umbilic. [6]

Q.6

(g) Consider the surface of revolution $\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$, where $f > 0$ and $f'^2 + g'^2 = 1$. Then prove the following statements. [6]
  (1) The curves $\overline{\alpha}(t) = \sigma(u(t), v_0)$ are geodesics.
  (2) A curve $\overline{\beta}(t) = \sigma(u_0, t)$ is a geodesic if and only if $f'(u_0) = 0$.

(h) Let σ be a surface patch of an oriented surface $\mathcal{S}$. Show that $L_v - M_u = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2$ and $M_v - N_u = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2$. [6]

OR

(h) Let $\mathcal{S}$ be a surface such that its Gaussian curvature is -1 everywhere. Show that the sum of interior angles of a triangle on $\mathcal{S}$ is less than π. State the result you use. [6]

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[42/A-16] **SEAT No.**_______

No. of printed pages: 2

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination

Tuesday, 18th April, 2017

PS02CMTH04, Functional Analysis - I

Time: 10:00 a.m. to 1:00 p.m. Maximum marks: 70

Note: All the questions are to be answered in answer book only. Figures to the right indicate full marks of the respective question. Assume standard notations wherever applicable.

Q-1 Choose the most appropriate option for each of the following questions: [8]

1. Let H be a Hilbert space and $x, y \in H$ be orthonormal. Then $\|x-y\|^2 =$ ____.
   (a) 4 (b) 2 (c) 0 (d) $\sqrt{2}$
2. Let $H = \mathbb{C}^2$ be Hilbert space over $\mathbb{C}$ and $x = (x_1, x_2)$, $y = (y_1, y_2) \in H$. Then $\langle x, y\rangle =$ ____ is an inner product on H.
   (a) $x_1\bar{y}_1 + 5x_2\bar{y}_2$ (b) $\bar{x}_1 y_1 + \bar{x}_2 y_2$ (c) $x_1\bar{x}_2 + y_1\bar{y}_2$ (d) $x_1\bar{y}_1 - x_2\bar{y}_2$
3. If E is a non-empty subset of a Hilbert space H and $x \in H$, then the number of best approximation from $E^{\perp}$ to x is ____.
   (a) 0 (b) 1 (c) 2 (d) infinite
4. Let H be a Hilbert space and $x_1, x_2, x_3 \in H$ be orthonormal. Then their Gram matrix ____.
   (a) is identity matrix (c) is singular
   (b) is a zero matrix (d) cannot be computed
5. Let H be a Hilbert space and $T \in BL(H)$ be isometry. Then ____ need not be true.
   (a) T is one-one (b) T is bounded below (c) T is onto (d) T^* is onto
6. Let H be a Hilbert space and $S, T \in BL(H)$ be self-adjoint. Then ____ is self-adjoint.
   (a) ST (b) $2T + 3iS$ (c) $2S + 3iT$ (d) $2S + 3T$
7. Let H be a Hilbert space and $T \in BL(H)$. Then ____.
   (a) $\sigma(T) \subset \sigma_a(T)$ (b) $\sigma_a(T) \subset \sigma_e(T)$ (c) $\sigma_a(T) \subset \overline{W(T)}$ (d) $\sigma_a(T) \subset W(T)$
8. Let H be a Hilbert space and $T \in BL(H)$ be non-zero compact and $0 \neq \lambda \in \sigma_e(T)$. Then $\ker(T - \lambda I)$ is ____.
   (a) $\{0\}$ (b) H (c) infinite dimensional (d) finite dimensional

Q-2 Attempt *Any Seven* of the following: [14]

(a) Let H be Hilbert space and $T \in BL(H)$. Define $\langle \cdot, \cdot \rangle_T$ by $\langle x, y\rangle_T = \langle Tx, Ty\rangle$, $x, y \in H$. Show that $\langle \cdot, \cdot \rangle_T$ is an inner product if T is one-one.

(b) State and prove Parallelogram law for an inner product space.

(c) Let H be a Hilbert space and $\phi \neq E \subset H$. Then show that $E^{\perp}$ is closed in H.

(d) Show by an example that, in a Hilbert space H, a weakly convergent sequence need not be convergent.

(e) Let H be a Hilbert space and $T \in BL(H)$. Show that $\ker(T) = \ker(T^*T)$.

(f) Let H be Hilbert space over $\mathbb{C}$ and $S \in BL(H)$. Show that there exists self-adjoint operators $A, B \in BL(H)$ such that $S = A + iB$.

(g) Define Hilbert-Schmidt operator.

(h) Let T be the right-shift operator on ℓ^2, i.e. $T(x(1), x(2), \ldots) = (0, x(1), x(2), \ldots)$, $(x(1), x(2), \ldots) \in \ell^2$. Show that $0 \in \sigma(T)$ but $0 \notin \sigma_a(T)$.

(i) Let H be a Hilbert space and $T \in BL(H)$. If $S : H \to H$ be linear and compact, then show that ST is compact.

Q-3 (a) State and prove Bessel's inequality. [6]

(b) Let X be an inner product space and E be an orthonormal subset of X. Show that for each $x \in X$, the set $E_x = \{u \in E : \langle x, u \rangle \neq 0\}$ is countable. [6]

OR

(b) State Gram-Schmidt orthonormalization theorem. Apply Gram-Schmidt process to orthonormalize the set $\{x_1, x_2, x_3, x_4\}$, where $x_1 = (1,0,0,0)$, $x_2 = (1,1,0,0)$, $x_3 = (1,1,1,0)$, $x_4 = (1,1,1,1) \in \mathbb{R}^4$. [6]

Q-4 (a) State and prove unique Hahn-Banach extension theorem. [6]

(b) Let $E \neq \phi$ be a closed convex subset of a Hilbert space H. Then show that E contains a unique element of minimum norm. [6]

OR

(b) State and prove Riesz-representation theorem. [6]

Q-5 (a) Let H be a Hilbert space and $T \in BL(H)$ be such that T^* is bounded below. Show that $R(T) = H$. [6]

(b) Let H be a Hilbert space and $T \in BL(H)$. Show that T is unitary if and only if T is an onto isometry. [6]

OR

(b) Let H be a Hilbert space and $T \in BL(H)$ be self-adjoint. Show that $\|T\| = \sup\{|\langle Tx, x \rangle| : x \in H, \ \|x\| \leq 1\}$. [6]

Q-6 (a) Let H be a Hilbert space and $T \in BL(H)$. Define numerical range $W(T)$ of T. If T is self-adjoint then show that $m_T \in \sigma_a(T)$, where $m_T = \inf\{\lambda : \lambda \in W(T)\}$. [6]

(b) Show that every Hilbert-Schmidt operator T on a separable Hilbert space H is compact. [6]

OR

(b) For a Hilbert space H, show that if $T \in BL(H)$ is compact then T^* is compact. [6]

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[20/A15]

SEAT No.________

No of printed pages: 2

Sardar Patel University

M.Sc. (Sem-II), PS02CMTH05, Methods of Partial Differential Equations;
Thursday, 20^{th} April, 2017; 10.00 a.m. to 01.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. The order of equation $(3 - D')(DD' + 2)z = 0$ is
   (A) 3 (B) 2 (C) 1 (D) 0
2. The equation $r + 2s + t = 0$ is same as $F(D, D')z = 0$ where $F(D, D')$ is
   (A) $(D^2 + 2D'^2)$ (B) $(D - D')^2$ (C) $(D + D')^2$ (D) none of these
3. The equation $xyDz = 5D'^2z$ classified as
   (A) elliptic (B) parabolic (C) hyperbolic (D) none of these
4. In Monge's method, the $\lambda-$ quadratic equation of $rt - s^2 + 1 = 0$ is
   (A) $\lambda^2 - 1 = 0$ (B) $\lambda^2 + 1 = 0$ (C) $\lambda^2 + 2 = 0$ (D) none of these
5. The relation between cylindrical and Cartesian coordinates are given by
   (A) $x = r\cos(-\theta), y = r\sin(-\theta), z = z$
   (B) $x = r\sin\theta, y = r\cos\theta, z = z$ (C) $x = r\cos(-\theta), y = r\sin\theta, z = z$
   (D) none of these
6. The two dimensional wave equation is
   (A) $u_{xx} + u_{yy} = 0$ (B) $u_{xx} + u_{yy} = \frac{1}{c^2}u_{tt}$
   (C) $u_{xx} = \frac{1}{k}u_{tt}$ (D) none of these
7. If u and v are any two solutions of Neumann BVP, then
   (A) $u - v = \alpha$ $(\alpha \in \mathbb{R})$ (B) $u = \alpha v$ $(1 \neq \alpha \in \mathbb{R})$
   (C) $u = v$ (D) none of these
8. If u and v are any two solutions of Dirichlet BVP, then
   (A) $u = \alpha v$ $(1 \neq \alpha \in \mathbb{R})$ (B) $u - v = \alpha$ $(0 \neq \alpha \in \mathbb{R})$
   (C) $u = v$ (D) none of these

Q.2 Attempt any *seven*: [14]

(a) Define general solution of partial differential equation.
(b) Find a pde by eliminating f and g from $z = f(x + 3y) + g(x - 3y)$.
(c) Solve: $(D^2 - DD' - 6D'^2)z = 0$.
(d) Classify the region in which the equation $(D^2 - 3yDD' + D'^2)z = 0$ is hyperbolic.
(e) Find $u = u(x, y)$ and $v = v(x, y)$ to covert $r = t$ in the canonical form.
(f) Find D^2z where $z = z(x, y)$ is transformed by $u = \log x$ and $v = \log y$.
(g) Write down wave equation in spherical coordinates.
(h) State Green's theorem.
(i) Define equipotential surface.

Q.3

(a) If $(\beta D' + \gamma)^2$ $(\beta \neq 0)$ is a factor of $F(D, D')$, then prove that $e^{-\frac{\gamma}{\beta}y}[\phi_1(\beta x) + y\phi_2(\beta x)]$ is a solution of $F(D, D')z = 0$, where ϕ_1 and ϕ_2 are arbitrary functions. [6]

(b) Find the general solution of $(D^2 + 2DD' - 8D'^2)z = (3y + 2x)^{\frac{1}{2}}$. [6]

OR

(b) Find the general solution of $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x + y)$.

Q.4

(a) Find canonical form of $r + x^2 t = 0$. [6]

(b) Solve: $3s + rt - s^2 = 2$ using Monge's method. [6]

OR

(b) Solve: $q^2 r - 2pqs + p^2 t = 0$ using Monge's method.

Q.5

(a) Solve: $(y^2 D'^2 - 2xyDD' + yD')z = xy$. [6]

(b) Solve $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{k}\frac{\partial \varphi}{\partial t}$ by the method of separation of variables and show that the solution can be put in the form of $\varphi(x, y, t) = e^{i(nx+my)-k(n^2+m^2)t}$, where n and m are constants. [6]

OR

(b) Solve $\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$ by the method of separation of variables and show that the solution can be put in the form $\psi(r, \theta, z) = J_n(mr)e^{(mz+in\theta)}$, where m is constant and J_n is a Bessel's function of order n.

Q.6

(a) State and prove maximum principle. [6]

(b) Find $u = u(x, y)$ such that $\nabla^2 u = 0$ in $\{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$ with [6]

$$\begin{aligned} u(x,0) &= f(x), \; 0 \leq x \leq a \\ u(a,y) &= 0, \quad 0 \leq y \leq b \\ u(x,b) &= 0, \quad 0 \leq x \leq a \\ u(0,y) &= 0, \quad 0 \leq y \leq b. \end{aligned}$$

OR

(b) Discuss the Neumann interior BVP for a circle.

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[C81] SEAT No.________

No. of printed pages: 2

SARDAR PATEL UNIVERSITY
M. Sc. (Semester II) Examination

Date: 22-04-2017 **Time: 10.00 To 01.00**
Subject: MATHEMATICS **Paper No. PS02EMTH02 – (Graph Theory – I)**
Total Marks: 70

1. Choose the correct option for each question: [8]

(1) The diameter of the graph $K_{1,n}$ $(n > 1)$ is
(a) 1 (b) 2 (c) n (d) n + 1

(2) A regular digraph is
(a) Euler (b) connected (c) balanced (d) none of these

(3) For $G = C_n$ with clockwise direction, rank(B) is
(a) 1 (b) $n - 1$ (c) n (d) none of these

(4) If G is a complete symmetric digraph with n vertices, then $|E(G)| =$
(a) $\frac{n(n-1)}{2}$ (b) n (c) $n(n-1)$ (d) n^2

(5) The coefficient c_5 in chromatic polynomial of P_5 is
(a) 5^5 (b) 5^2 (c) 5 (d) 5!

(6) Which of the following graphs is not Hamiltonian?
(a) K_n (b) $K_{n,n}$ (c) P_n (d) C_n

(7) Let G be a simple graph without isolated vertex. Then a matching M in G is
(a) maximum $\Rightarrow$ perfect (c) maximal $\Rightarrow$ maximum
(b) maximum $\Rightarrow$ maximal (d) maximal $\Rightarrow$ perfect

(8) If $G = P_7$, then $(\alpha(G), \beta(G)) =$
(a) (3, 3) (b) (4, 4) (c) (4, 3) (d) (3,4)

2. Attempt any SEVEN: [14]

(a) Prove: If $K_{m,n} = K_{m+n}$, then $m = n = 1$.
(b) Prove or disprove: A balanced digraph is an Euler digraph.
(c) Define incidence matrix in a digraph.
(d) Define fundamental circuit matrix in a digraph.
(e) Prove: If G is a bipartite graph, then $\chi(G) = 2$.
(f) What is Four color problem?
(g) Prove or disprove: A minimal vertex cover is an independent set.
(h) Prove or disprove: The graph C_4 is isomorphic to $K_{2,2}$.
(i) Prove: If K_n has a perfect matching, then n is even.

3. (a) Define the following digraphs with examples: [6]
(i) Asymmetric (ii) complete asymmetric (iii) Symmetric (iv) complete symmetric.

(b) Let G be a digraph such that either for every $v \in V(G)$, $d^+(v) > 0$ or for every $v \in V(G)$, $d^-(v) > 0$. Prove that G has a directed circuit. [6]

OR

(b) Obtain De Bruijn cycle for $r = 3$ with all detail. [6]

4. (a) Let A and B denote resp. the incidence matrix and circuit matrix of a digraph G without self-loop. Then prove that $AB^T = 0$. [6]

(b) Define the following with examples: [6]
(i) out-tree & spanning out-tree (ii) in-tree & spanning in-tree.

OR

(b) Prove: An arborescence is a tree in which every vertex other than the root has an in-degree exactly one. [6]

5. (a) Prove: If G is a simple graph with n vertices & $2\delta(G) \geq n \geq 3$, then G is Hamiltonian [6]

(b) Let G be a k-chromatic graph with n vertices. Prove that $n \leq k\,\alpha(G)$. [6]

OR

(b) Define chromatic number $\chi(G)$ of a graph G. Give an example of a non-complete graph G with $\chi(G) = \Delta(G) + 1$. [6]

6. (a) Prove: If G is a bipartite graph, then $\alpha'(G) = \beta(G)$. [6]

(b) Prove: A matching M in graph G is maximum if and only if G has no M-augmenting path. [6]

OR

(b) Define $\alpha'(G)$, $\beta'(G)$ and find it with the corresponding sets for $G = K_{n,m}$ $(n \neq m)$. [6]

x-x-x-x-x-x

[9] A-8

SEAT No.________

No. of printed pages: 2

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - II Examination
Saturday, 22nd April, 2017
PS02EMTH04, Mathematical Classical Mechanics

Time: 10:00 a.m. to 1:00 p.m. Maximum marks: 70

Note: All the questions are to be answered in answer book only. Figures to the right indicate full marks of the respective question. Assume standard notations wherever applicable.

Q-1 Choose the most appropriate option for each of the following questions: [8]

1. Motion of a particle outside a sphere is _____ constraint.
   (a) a holonomic (b) a non-holonomic (c) a rheonomic (d) not a
2. Degrees of freedom of a particle moving on a plane given by $y = 3z + 5$ is _____.
   (a) 0 (b) 1 (c) 2 (d) 3
3. The curve of shortest distance between two points in space is a _____.
   (a) great circle (b) straight line (c) catenary (d) cycloid
4. If Lagrangian L does not depend on a generalized coordinated q_j explicitly then _____ is conserved.
   (a) Lagrangian (c) total energy
   (b) Hamiltonian (d) generalized momentum p_j
5. If all coordinates are cyclic then Routhian $R =$ _____.
   (a) L (b) $-L$ (c) H (d) $-H$
6. Which one of the following is correct?
   (a) $\frac{\partial H}{\partial p_j} = -q_j$ (b) $\frac{\partial H}{\partial p_j} = -\dot{q}_j$ (c) $\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$ (d) $\frac{\partial L}{\partial t} = -\frac{dH}{dt}$
7. Pick up the incorrect statement from the following:
   (a) A symplectic matrix is singular
   (b) Jacobian matrix for a canonical transformation is symplectic
   (c) Identity matrix is symplectic
   (d) Canonical transformations are invertible
8. $\{p_2, q_1\} =$ _____; notations being usual.
   (a) 0 (b) 1 (c) -1 (d) p_2q_1

Q-2 Attempt *Any Seven* of the following: [14]

(a) State the constraints for the motion of a simple pendulum.

(b) State Lagrange's equations of motion when conservative force is present.

(c) State the condition for extremum of $I = \int_{x_1}^{x_2} f(y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n, x)\, dx$.

(d) Define cyclic coordinate.

(e) State Hamilton's equations of motion in matrix form.

(f) State Hamilton's modified principle.

(g) If H denotes the Hamiltonian of a system then show that H is a constant of motion if and only if H does not depend on t explicitly.

(h) State the transformation equations for a generating function of type F_3.

(i) Using Poisson brackets, show that the following transformation is canonical: $Q = \frac{1}{p}$, $P = qp^2$.

Q-3 (a) Using D'Alembert's principle, derive Lagrange's equations of motion in the form $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$, $j = 1, 2, \ldots, n$. [6]

(b) What is simple harmonic oscillator? Obtain Lagrange's equation of motion for a simple harmonic oscillator. [6]

OR

(b) Show that $L' = L + \frac{dF(q_1, q_2, \ldots, q_n, t)}{dt}$ also satisfies Lagrange's equations of motion, where L is given Lagrangian of a system and F is an arbitrary differentiable function of its arguments. [6]

Q-4 (a) Using calculus of variations, determine the curve of shortest distance between two points in a plane. [6]

(b) The Lagrangian of a system is given by [6]
$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\,cos\theta.$
Obtain all generalized momenta. Which of them are conserved? Justify your answer.

OR

(b) If constraints are sceleronomic, potential is velocity independent and Lagrangian does not depend on t explicitly then show that the total energy of the system is conserved. [6]

Q-5 (a) Using Legendre transformation derive Lagrange's equations of motion from Hamilton's equations of motion. [6]

(b) Lagrangian of a 2-dimensional isotropic oscillator is given by [6]
$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)$. Obtain the corresponding Hamiltonian.

OR

(b) Derive Hamilton's equations of motion from Hamilton's modified principle. [6]

Q-6 (a) Show that Poisson brackets of two constants of motion is also a constant of motion. [6]

(b) Define canonical transformation. Check whether the transformation $Q = q\tan p$ and $P = \log\sin p$ is canonical or not. [6]

OR

(b) Define Poisson bracket. Compute $[u, v]$ and $[[u, v], v]$ for $u = q$ and $v = \frac{p^2}{2m} - mcq$, where c is constant, q is generalized coordinate and p is generalized momentum conjugate to q. [6]

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