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[12]

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - IV Examination

Wednesday, 25th April, 2018

PS04EMTH31, Algebra-II

Time: 02:00 p.m. to 05:00 p.m. Maximum marks: 70

Note: Figures to the right indicate full marks of the respective questions. Assume standard notations wherever applicable.

Q-1 Write the question number and appropriate option number only for each question. [8]

(a) _____ group is a module over $\mathbb{Z}$.

(i) Abelian (ii) *Finite* (iii) *Infinite* (iv) *Allofthese*

(b) _____ is a module over a division ring.

(i) Group (ii) Vector space (iii) Ring (iv) $\mathbb{Z}$

(c) $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(n) =$ _____, $(n \in \mathbb{Z})$ is a module homomorphism.

(i) $2n+1$ (ii) n^2+n (iii) $n-2$ (iv) $4n$

(d) Let R be a commutative ring, M be an R-module and $a \in M$. Then annihilator of $\{a\}$ is _____.

(i) $R \setminus \{0\}$ (ii) ideal of R (iii) R (iv) $\{0\}$

(e) The sequence $M \xrightarrow{f} N \longrightarrow 0$ is exact iff f is _____.

(i) surjective (ii) injective (iii) bijective (iv) constant

(f) _____ is not a free $\mathbb{Z}$-module.

(i) $\mathbb{Z}$ (ii) $4Z$ (iii) $3Z$ (iv) $\mathbb{Q}$

(g) If M is a finitely generated module over PID R, then $M =$ _____ $\oplus F$ for some free module of finite rank.

(i) $M^{\perp}$ (ii) M (iii) $T(M)$ (iv) none of these

(h) _____ is not a projective $\mathbb{Z}$-module.

(i) $\mathbb{Z} \times \mathbb{Z}$ (ii) $\mathbb{Q}$ (iii) $\mathbb{Z}$ (iv) none of these

Q-2 Attempt *Any Seven* of the following: [14]

(a) Define a left module over a ring R.

(b) Let R be a commutative ring and I be an ideal of I. Show that R/I is an R-module.

(c) Show that $\{(a,0) : a \in \mathbb{Z}\}$ and $\{(0,a) : a \in \mathbb{Z}\}$ are not isomorphic as $\mathbb{Z}^2$-modules.

(d) Show that $\mathbb{Z}$ cannot be written as a direct sum of its two proper submodules..

(e) Show $\mathbb{Z}[x]$ is not a finitely generated $\mathbb{Z}$-module.

(f) Define a *torsion free module.*

(g) Give an example of a split exact sequence. Justify your claim.

(h) Define and give an example of a *projective module.*

(i) Show that a free module over an integral domain is torsion free.

[P.T.O.]

Q-3 (j) State and prove First Isomorphism Theorem for modules. [6]

(k) Prove or disprove: $A \in M_2(\mathbb{R}) \mapsto A^t \in M_2(\mathbb{R})$

(i) is an $M_2(\mathbb{R})$-homomorphism.

(ii) is an $\mathbb{R}$-homomorphism.

OR

(k) If a sequence $M_1 \xrightarrow{\varphi} M \xrightarrow{\psi} M_2 \longrightarrow 0$ of R-module homomorphisms is exact, then show $\ker \varphi^* = \operatorname{Im} \psi^*$. [6]

Q-4 (l) State and prove the correspondence theorem for quotient of a module. [6]

(m) Define a *torsion module* and for a module show that $M/T(M)$ is torsion free. [6]

OR

(m) Let R be an integral domain and $x \in R \setminus \{0\}$. Show that $R \simeq Rx$ as R-modules but $R \simeq Rx$ as rings if and only if x is a unit in R. [6]

Q-5 (n) Show that for a family $\{M_i : i \in I\}$ of submodules of M, $M = \bigoplus_{i\in I} M_i$ if and only if each $x \in M$ can be expressed uniquely as $x = \sum_{i\in I} x_i$, where $x_i \in M_i$ and $x_i \neq 0$ for finitely many indices i. [6]

(o) Define *external and internal direct sums* of modules. Show that every internal direct sum is an external direct sum. [6]

OR

(o) Giving all details show that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \simeq \mathbb{Z}_d$, where d is the $d = \gcd(m, n)$ [6]

Q-6 (p) Let B be a basis of a free R-module M and $\varphi : B \to M$ be any mapping. Show that there exists a unique homomorphism $f : M \to M$ such that $f(x) = \varphi(x)$ for all $x \in B$. [6]

(q) Let M be a module over a PID R and $\operatorname{rank}(M) = n$. If $\{x_1, x_2, x_3, \ldots, x_n\}$ generates M, then show that X is a basis. [6]

OR

(q) Let M be a module over a division ring D of finite rank n and let $X = \{x_1, x_2, x_3, \ldots, x_n\} \subset M \setminus \{0\}$. If X is linearly independent, then show that it is a basis of M. [6]

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