SEAT No.______ **No. of Printed Pages : 2**

[26]

Sardar Patel University

M.Sc.(Mathematics)(Sem-IV); Examination 2018;
PS04EMTH27: Banach Algebras;
23-04-2018; Monday; Time-2.00 pm to 5.00 pm; Maximum Marks 70

Note: Notations and terminologies are standard. Throughout $\mathcal{A}$ is a unital Banach algebra.

Q.1 Choose correct option from given four choices. [08]

(i) Which of the following function belongs to $C^1([0,1])$?

(a) $f(t) = t^{\frac{2}{3}}$ (b) $f(t) = t^{\frac{3}{2}}$ (c) $f(t) = t^{\frac{1}{3}}$ (d) $f(t) = t^{\frac{1}{2}}$

(ii) Let $x \in \mathcal{A}$. Then which of the following statements is true?
(a) If x has left inverse, then it has a right inverse;
(b) If x has a left inverse and a right inverse, then x is invertible;
(c) The x has at most one right inverse;
(d) The x has at most one left inverse.

(iii) Let $x = (-1, 0, 1) \in \mathbb{C}^3$. Then the spectrum $\sigma(x) =$

(a) $\{-1, 0, 1\}$ (b) $\{-1, 1\}$ (c) $\{0\}$ (d) $[-1, 1]$

(iv) Let $f(z) = z^{\frac{5}{2}}$ ($z \in \mathbb{D}$). Then the spectral radius $r(f)$ of f in $A(\mathbb{D})$ is

(a) 1 (b) 2 (c) 5 (d) $\frac{5}{2}$

(v) Let $\mathcal{A}$ be commutative, $\varphi \in \Delta(\mathcal{A})$, and $\ker\varphi$ be kernel of φ. Which of following is true?

(a) Every element of $\ker\varphi$ is invertible;
(b) Atleast one element of $\ker\varphi$ is invertible;
(c) Every element of $\ker\varphi$ is singular;
(d) Every element of $\ker\varphi$ is a zero devisor.

(vi) Let $\mathcal{A}$ be commutative. Then the Gel'fand space $\Delta(\mathcal{A})$ with the Gel'fand topology is a

(a) normal space;
(b) Banach space
(c) metric space;
(d) normed linear space;

(vii) Which of the following is not a Banach $*$-algebra with natural involution $f^*(x) = \overline{f(x)}$?

(a) $C(X)$ (b) $C^1[0,1]$ (c) $C^2[0,1]$ (d) $A(\mathbb{D})$

(viii) Let $\mathcal{A}$ be a commutative C^*-algebra and $x \in \mathcal{A}$ such that $x^* = x$. Then the spectrum $\sigma(x)$ of x is contained in

(a) $[0,1]$ (b) $[-1,1]$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}$

Q.2 Attempt any seven [14]

(i) Show that the set G of invertible elements is a group with ring multiplication.
(ii) Prove that the set S of singular elements is closed in $\mathcal{A}$.
(iii) Define the spectrum $\sigma(x)$ and the spectral radius $r(x)$ of an element x in $\mathcal{A}$.
(iv) Let $x, y \in \mathcal{A}$ such that $xy = yx$. Prove that $r(xy) \leq r(x)r(y)$.
(v) State and prove Gel'fand Mazure theorem for a Banach algebra which is a division algebra.
(vi) Let $r \in \mathcal{A}$ such that $1 - ar$ is invertible for every $a \in \mathcal{A}$. Prove that $r \in Rad(\mathcal{A})$.
(vii) Prove that every complex homomorphism on $\mathcal{A}$ is continuous.
(viii) Let $\mathcal{A}$ be commutative and $x \in \mathcal{A}$. Prove that $\widehat{x} : \Delta(\mathcal{A}) \longrightarrow \mathbb{C}$ is continuous.
(ix) Define a Banach $*$-algebra and a C^*-algebra.

Q.3

(a) Let S be the set of all singular elements and Z be the set of all topological divisor of zero in $\mathcal{A}$. Prove that $bd(S) \subset Z \subset S$. [6]

(b) Define the set $l^1(\mathbb{Z})$ as well as all operations and a norm on $l^1(\mathbb{Z})$ to make it a Banach algebra. [6]

OR

(b) Prove that the set G of all invertible element is open in $\mathcal{A}$ and the mapping $g : G \longrightarrow G$ defined as $g(x) = x^{-1}$ is a homeomorphism. [6]

Q.4

(a) Prove that the spectrum $\sigma(x)$ is a non-empty, compact subset of $\mathbb{C}$. [6]

(b) Prove that $\inf\{\|x^n\|^{\frac{1}{n}} : n \in \mathbb{N}\} = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ $(x \in \mathcal{A})$. [6]

OR

(b) Define the radical Rad$(\mathcal{A})$. Prove that it is a closed, two sided ideal in $\mathcal{A}$. State all results explicitly used in the proof. [6]

Q.5

(a) Let X be a compact T_2 space. Prove that $\Delta(C(X)) \cong X$. [6]

(b) Let $\mathcal{A}$ be commutative, $\Delta(\mathcal{A})$ be the Gelfand space of $\mathcal{A}$, and $Max(\mathcal{A})$ be the space of all maximal ideals in $\mathcal{A}$. Define $\Lambda : \Delta(\mathcal{A}) \longrightarrow Max(\mathcal{A})$ as $\Lambda(\varphi) = \ker\varphi$. Prove that Λ is a bijective map. [6]

OR

(b) Give sketch of the proof to show that $\Delta(\mathcal{A})$ is a contact, Hausdorff space. [6]

Q.6

(a) State and prove the Gel'fand-Neumark Theorem for a commutative C^*-algebra. [6]

(b) State and prove Banach-Stone Theorem. [6]

OR

(b) Let X be a compact, T_2-space and I be a proper, closed ideal in $C(X)$. Prove that there is a nonempty, closed set $E \subset X$ such that $I = I(E) = \{f \in C(X) : f(x) = 0 \ (x \in E)\}$. [6]

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