SEAT No.________

[89]

No of printed pages: 2

Sardar Patel University

M.Sc. (Sem-IV), PS04EMTH22, Mathematical Probability Theory;
Friday, 13^{th} April, 2018; 02.00 p.m. to 05.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. If $X = I_A$ and $P(A) = \frac{1}{2}$, then the the set of discontinuities of distribution function of X, is
   (A) ϕ (B) $\{\frac{1}{2}\}$ (C) $\{0, \frac{1}{2}\}$ (D) $\{0, 1\}$
2. If $P(A) = \frac{1}{2} = P(B)$ and $P(A^c \cap B^c) = \frac{1}{3}$, then $P(A^c \cap B)$ is
   (A) $\frac{5}{6}$ (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$
3. If $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{L} Y$, where $Y_n, Y \neq 0$, then which is true from following ?
   (A) $X_n + Y_n \xrightarrow{L} X + Y$ (B) $X_n Y_n \xrightarrow{L} XY$
   (C) $\frac{X_n}{Y_n} \xrightarrow{L} \frac{X}{Y}$ (D) none of (A),(B),(C) is true
4. If $F_n \to F$ weakly and F_n are continuous on $\mathbb{R}$, then
   (A) $\int F_n \to \int F$ (B) F is continuous on $\mathbb{R}$
   (C) $F_n' \to F'$ (D) none of (A),(B),(C) is true
5. If $\phi(u)$ is characteristic function of random variable X, then
   (A) $\phi(u) \leq \phi(0)$ (B) $\phi(u) > \phi(0)$
   (C) $\phi(-u) = -\overline{\phi}(u)$ (D) none of (A),(B),(C) is true
6. If ϕ is characteristic function of random variable X, then $---$ is also characteristic function.
   (A) $|\phi|^2$ (B) $|\phi|$ (C) $|\phi|^3$ (D) all these are true
7. The standard normal random variable having (mean, variance)is
   (A) $(1, 0)$ (B) $(0, 0)$ (C) $(0, 1)$ (D) $(1, 1)$
8. Which inequality used for proving Strong Law of Large Numbers ?
   (A) Holder's inequality (B) Jenson's inequality
   (C) Chebyshev's inequality (D) Minkowski's inequality

Q.2 Attempt any ***seven***: [14]

(a) Define probability space.

(b) Let $\Omega = \{a, b, c, d\}, \mathcal{A} = \{\phi, \Omega, \{a, b\}, \{c, d\}\}$ and $X : \Omega \to \mathbb{R}$ defined by $X(a) = -1 = X(b), X(c) = 1, X(d) = -2$. Is X a random variable?

(c) Define convergence in probability.

(d) Define weak convergence.

(e) If $X_n \xrightarrow{L} X$, where $F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1-x)^n & , 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$. Find F_X.

(f) Let X be a r.v. having pdf, $f_X(x) = \begin{cases} e^{-x} & , x \geq 0 \\ 0 & , x < 0. \end{cases}$. Find the corresponding characteristic function.

(g) State Inversion theorem for characteristic function.

(h) State Weak Law of Large Numbers.

(i) What is Kolmogorav's inequality ?

(P.T.O.)

Q.3

(a) Let f be a continuous function on $\mathbb{R}$ and $X_n \xrightarrow{P} X$. Then show that $f(X_n) \xrightarrow{P} f(X)$. [6]

(b) Let $Z = (X, Y)$ be vector random variable. Prove: $Z^{-1}(\mathcal{B}_2) = \sigma\left(X^{-1}(\mathcal{B}) \cup Y^{-1}(\mathcal{B})\right)$, where $\mathcal{B}_2$ and $\mathcal{B}$ are Borel σ-algebras in $\mathbb{R}^2$ and $\mathbb{R}$ respectively. [6]

OR

(b) Two random variables X and Y having the joint pdf

$$f(x,y) = \begin{cases} cxy, & x \geq 0,\ y \geq 0\ , x+y \leq 1 \\ 0, & otherwise \end{cases}$$

Find (i) c (ii) marginal pdf of X (iii) marginal pdf of Y.

Q.4

(a) If $X_n \xrightarrow{P} X$ then show that $X_n \xrightarrow{L} X$. What about converse ? [6]

(b) State and prove Jordan Decomposition Theorem. [6]

OR

(b) If $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{L} c$, then show that $X_n + Y_n \xrightarrow{L} X + c$.

Q.5

(a) State and prove weak compactness theorem. [6]

(b) Prove that every characteristic function is uniformly continuous on $\mathbb{R}$. State results which you use. [6]

OR

(b) Let X be a continuous r.v. with characteristic function $\phi(u)$. Then show that $E(X^k) = \frac{1}{i^k}\phi^k(0)$, where $\phi^k(u)$ is k-th derivative of ϕ with respect to u.

Q.6

(a) State and prove Strong Law of Large Numbers. [6]

(b) State and prove Central Limit Theorem. [6]

OR

(b) Let $\{X_n\}$ be independent random variables with $P(X_n = \pm k^{\lambda}) = \frac{1}{2}$, where $k, \lambda > 0$. Then show that the Weak Law of Large Numbers holds iff $0 < \lambda < \frac{1}{2}$.

—X—