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SEAT No.________

[87]

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Sardar Patel University

Mathematics

M.Sc. Semester IV

Friday, 13 April 2018

2.00 p.m. to 5.00 p.m.

PS04EMTH01 - Problems and Exercises in Mathematics III

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) Let $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,\frac{1}{2}]}$, $f_2 = \chi_{[\frac{1}{2},1]}$, $f_4 = \chi_{[0,\frac{1}{3}]}$, $\cdots$ Then $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$ equals

(a) 0 (b) 1 (c) ∞ (d) $\frac{1}{2}$

(2) Let f be the Cantor function on $[0,1]$. Then the total variation of f is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

(3) Let $f : \mathbb{D} \to \mathbb{D}$ be bijective and analytic. If $f(0) = 0$, then

(a) $f'(0) = 1$ (b) $|f'(0)| = 1$ (c) $|f'(0)| < 1$ (d) $|f'(0)| > 1$

(4) 0 is _____ of $\frac{e^{-z^2}}{z^2}$

(a) a pole
(b) a removable singularity
(c) an essential singularity
(d) a nonisolated singularity

(5) Unit circle S^1 is homeomorphic to _____.

(a) $[0,1)$ (b) $(0,1)$ (c) $[0,1]$ (d) none of these

(6) _____ is not a topological property.

(a) Being Hausdorff
(b) Compactness
(c) Completeness
(d) Path-connectedness

(7) If $A = \{x \in \mathbb{F}_{2^5} \mid x^{31} = 1 \text{ and } x^k \neq 1 \text{ for all } 0 < k \leq 30\}$, then the number of elements in A is _____.

(a) 1 (b) 30 (c) 31 (d) 32

(8) Which of the following quotient ring is a field?

(a) $\mathbb{Z}_3[x]/\langle x^2 - x + 1\rangle$
(b) $\mathbb{Z}_2[x]/\langle x^4 + x^2 + 1\rangle$
(c) $\mathbb{Q}[x]/\langle x^3 + x^2 + x + 1\rangle$
(d) $\mathbb{Q}[x]/\langle x^4 + x^3 + x^2 + x + 1\rangle$

Q.2 Attempt any *Seven*. [14]

(a) Show that every Lipschitz function on $[a,b]$ is absolutely continuous. Is the converse true? Why?

(b) Let f_n and f be measurable function on a measurable set E. If $f_n \to f$ uniformly on E, then show that $f_n \to f$ in measure on E.

(c) Let $f \in BV[a,b]$, and let $f(x) \neq 0$ for any $x \in [a,b]$. Will $\frac{1}{f}$ be in $BV[a,b]$? Why?

(d) If f is an entire function and $|f(z)| \leq |e^z|$ for all $z \in \mathbb{C}$, then find a formula for f.

(e) Let p be a nonconstant polynomial. Show that $p(z_n) \to \infty$ whenever $z_n \to \infty$

(f) Define order topology.

(g) Give an example of non-compact sets A and B such that $A \cup B$ and $A \cap B$ is compact.

(h) Construct fields of order 16 and 125.

(i) Determine the number of monic quadratic reducible polynomials in $\mathbb{Z}_p[x]$.

Q.3

(a) Let $f, g : [0,1] \to \mathbb{R}$ be $f(x) = x^2 \sin(\frac{1}{x})$ and $g(x) = x \sin(\frac{1}{x^2})$ if $x \neq 0$ and $f(0) = g(0) = 0$. Show that f is of bounded variation but g is not. [6]

(b) Maximize the product $x_1 x_2 \cdots x_n$ subject to $x_i > 0$ for all i and $x_1 + x_2 + \cdots + x_n = n$. Hence prove that the geometric mean of n positive real numbers is less than or equal to their arithmetic mean. [6]

OR

(b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and vanishing at infinity, then show that f is uniformly continuous. Also, prove that there is $x_0 \in \mathbb{R}$ such that $|f(x_0)| = \sup\{|f(x)| : x \in \mathbb{R}\}$. [6]

Q.4

(c) Suppose that f and g are analytic on an open set containing $\overline{B(0;R)}$ and both f are g are does not vanish at any point of $\overline{B(0;R)}$. If $|f(z)| = |g(z)|$ for all z with $|z| = R$, then show that there is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f = \lambda g$. [6]

(d) Let f be analytic on $B(0 : R)$ and have the representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in B(0; R)$. For $n \in \mathbb{N}$, let $S_n(z) = \sum_{k=0}^{n} a_k z^k$. Show that $S_n \to f$ uniformly on every compact subset of $B(0; R)$. State the results you use. [6]

OR

(d) Let f be analytic on an open set containing $\overline{\mathbb{D}}$. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{D}$ and $z_1, z_2, \ldots, z_n$ are all zeros of f in $\mathbb{D}$. Show that $|f(z)| \leq M \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - \overline{z_k} z} \right|$ for all $z \in \mathbb{D}$. Also, find the formula for f if $f(0) \neq 0$ and $f(0) = M e^{i\theta} z_1 z_2 \cdots z_n$ for some $\theta \in \mathbb{R}$. [6]

Q.5

(e) Let X be a topological space and $x, y \in X$. Define a path in X from x to y. Show that a path is always connected but a connected space need not be path-connected. [6]

(f) Let τ_f and τ_c denote the co-finite topology and the co-countable topology on $\mathbb{R}$ respectively. Is $[0,1]$ compact in τ_f? Is it compact in τ_c? Justify. [6]

OR

(f) Let $A = (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{Q} \times \mathbb{Q})$ and $B \subset \mathbb{R} \times \mathbb{R}$ be the set with at least one rational coordinate. Determine whether A and B are connected or not. [6]

Q.6

(g) Show that the polynomial $x^p - px + 1$ is irreducible over $\mathbb{Q}$ for all primes p. Also determine whether $x^5 + x^2 + 1$ is irreducible over $\mathbb{Q}$ or not. [6]

(h) For distinct primes p and q, show that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$. Does there exist a field isomorphism from $\mathbb{Q}(\sqrt{p})$ to $\mathbb{Q}(\sqrt{q})$? Justify. [6]

OR

(h) Compute the Galois group of the polynomial $x^3 - 2$ over $\mathbb{Q}$. [6]

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