SEAT No.________

[26 & A-4]

No of printed pages: 2

Sardar Patel University
Mathematics
M.Sc. Semester IV
Thursday, 02 November 2017
10.00 a.m. to 1.00 p.m.
PS04CMTH01 - Complex Analysis II

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Which of the following is an open map?
(a) $\sinh z$ (b) $e^{|z|}$ (c) $|z|$ (d) $e^{-|z|}$

(2) Let $\gamma(t) = 2e^{it}$, $0 \le t \le 4\pi$. Then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z+3} =$ _____
(a) 2 (b) 3 (c) $\frac{2}{3}$ (d) none of these

(3) Let $R = \{z \in \mathbb{C} : |z - 3i| \le 1\}$, and let $f(z) = z$. Then the minimum value of $|f|$ on R is
(a) 1 (b) 2 (c) 3 (d) $3i$

(4) Let p be a polynomial of degree n. Suppose that all the zeros of p are contained in $\{z \in \mathbb{C} : |z| < 2\}$. If $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$, then the value of $\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}dz$ is
(a) 0 (b) n (c) $n\pi i$ (d) $2n\pi i$

(5) Let $\mathbb{H}^+$ and $\mathbb{H}_-$ be open upper half plane and open lower half plane respectively. Which of the following are not conformally equivalent?
(a) $\mathbb{D}$ and $\mathbb{C}$ (b) $\mathbb{D}$ and $\mathbb{H}^+$ (c) $\mathbb{D}$ and $\mathbb{H}_-$ (d) $\mathbb{H}^+$ and $\mathbb{H}_-$

(6) Let $f(z) = \frac{1-\cos z}{z^4}$. Then 0 is _____ of f.
(a) a removable singularity (c) a pole of order 2
(b) a pole of order 1 (d) an essential singularity

(7) Let (z_n) be a sequence of a complex numbers such that $\prod_{n=1}^{\infty} z_n = \ell$ and $\ell \neq 0$. Then $\lim_{n\to\infty} z_n$ is _____.
(a) 0 (b) 1 (c) ℓ (d) $\frac{1}{\ell}$

(8) If f is a nonconstant entire function, then the set of zeros of f is
(a) countably infinite (b) countable (c) finite (d) uncountable

Q.2 Attempt any ***Seven***. [14]

(a) If $n \in \mathbb{N}$, then evaluate $\int_\gamma \frac{1}{(z-1)^n}dz$, where $\gamma(t) = 1 + e^{it}$, $0 \le t \le 2\pi$.
(b) State Cauchy Integral Formula (First Version).
(c) Let G be an open connected set. If $f, g : G \to \mathbb{C}$ are analytic, g is a nonzero map and $fg = 0$, then show that $f = 0$.
(d) Let $a \in \mathbb{C}$ with $|a| > e$, and let $n \in \mathbb{N}$. Show that the equation $e^z - az^n = 0$ has n solutions in $|z| = 1$.
(e) Does there exist an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$? Why?
(f) Define a space $C(G, \Omega)$ and define a metric on it.
(g) If p is a polynomial of degree m, then show that ∞ is a pole of p order m.
(h) Discuss the convergence of $\prod_{n=1}^{\infty}(1 + \frac{1}{n})$.
(i) If $u : \overline{B(a;r)} \to \mathbb{R}$ is harmonic, then show that $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$.

Q.3

(a) Let γ be a rectifiable curve. Let φ be a function defined and continuous on $\{\gamma\}$. For each $m \in \mathbb{N}$, let $F_m(z) = \int_\gamma \varphi(w)(z-w)^{-m}dw$, $z \notin \{\gamma\}$. Show that each F_m is analytic in $\mathbb{C} - \{\gamma\}$ and that $F'_m(z) = mF_{m+1}(z)$ for every $z \in \mathbb{C} - \{\gamma\}$. [6]

(b) Let G be a region. Let $f : G \to \mathbb{C}$ be continuous function such that $\int_T f(z)dz = 0$ for every triangular path T in G. Show that f is analytic in G. [6]

OR

(b) If γ is a rectifiable curve and f is a continuous map on $\{\gamma\}$, then show that $\int_\gamma f = -\int_{-\gamma} f$ and $|\int_\gamma f| \leq \int_\gamma |f||dz| \leq V(\gamma)\sup[|f(z)| : z \in \{\gamma\}]$. [6]

Q.4

(c) Let G be an open connected subset of $\mathbb{C}$, and let $f : G \to \mathbb{C}$ be analytic. If the set $\{z \in G : f(z) = 0\}$ has a limit point in G, then show that there is $a \in G$ such that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. [6]

(d) When is a function $f : [a, b] \to \mathbb{R}$ called convex? Show that a differentiable function $f : [a, b] \to \mathbb{R}$ is convex if and only if f' is an increasing function. [6]

OR

(d) State Schwarz's Lemma. Suppose that f is analytic on the open unit disc $\mathbb{D}$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. If $a \in \mathbb{D}$ and $f(a) = \alpha$, then show $|f'(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2}$. Also, show that if $|f'(a)| = \frac{1-|\alpha|^2}{1-|a|^2}$, then there is $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = \varphi_{-\alpha}(c\varphi_a(z))$ for all $z \in \mathbb{D}$. [6] [6]

Q.5

(e) If (f_n) is a sequence in $H(G)$ and f belongs to $C(G, \mathbb{C})$ such that $f_n \to f$, then show that f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$. [6]

(f) Let G be an open subset of $\mathbb{C}$, and let $a \in G$. Let f be analytic on $G - \{a\}$, and let a be singularity of f. Show that a is a pole of f if and only if there is an analytic function $g : G \to \mathbb{C}$ with $g(a) \neq 0$ and $m \in \mathbb{N}$ such that $f(z) = \frac{g(z)}{(z-a)^m}$ for every $z \in G - \{a\}$. [6]

OR

(f) When is a subset $\mathscr{F}$ of $H(G)$ called locally bounded? Show that a subset $\mathscr{F}$ of $H(G)$ is locally bounded if and only if for each compact set $K \subset G$ there is a constant M such that $|f(z)| \leq M$ for all f in $\mathscr{F}$ and z in K. [6]

Q.6

(g) State *Weierstrass Factorization Theorem.* Hence find a canonical factorization of $\sinh \pi z$. [6]

(h) Let f be analytic on an open set containing $\overline{B(0; r)}$. If f has exactly one simple zero $a = re^{i\alpha}$ on $|z| = r$, then show that $\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|d\theta$. [6]

OR

(h) Let f be analytic within and on a circle $|z| = R$. If $0 \leq r < R$, then show that [6]

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\varphi})}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}d\varphi.$$

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(A-22) SEAT No.________ No of printed pages: 2

Sardar Patel University
Mathematics
M.Sc. Semester IV
Friday, 10 November 2017
10.00 a.m. to 1.00 p.m.
PS04EMTH01 - Problems and Exercises in Mathematics III

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [8]

(1) $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = \dots\dots\dots$

(a) 1 (b) 2 (c) 4 (d) none of these

(2) Let $w^3 = 1$ and $w \neq 1$. Then the value of $1 + w^2 + w^4$ is

(a) 0 (b) 1 (c) w (d) w^2

(3) Let R be the closed triangular region whose vertices are $0, 1$ and i. Then the maximum of value of $|z+1|$ over R is

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

(4) Let $f(z) = \frac{\sin z}{z}$. Then 0 is .... of f.

(a) a removable singularity
(b) a pole
(c) an essential singularity
(d) a non isolated singularity

(5) Let E be a subset of $\mathbb{R}$. Then the set of discontinuity of $\chi_E : \mathbb{R} \to \mathbb{R}$ is

(a) E (b) ∂E (c) E° (d) $\overline{E}$

(6) Let $f, g \in C[0,1]$ and that $fg = 0$. Which of the following is true?

(a) $f = 0$ or $g = 0$
(b) If $f \neq 0$, then $g = 0$
(c) $f = 0$ and $g = 0$
(d) None of these

(7) If $A = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$, then the value of $d(A, B)$ is

(a) 0 (b) 1 (c) 2 (d) ∞

(8) One point compactification of $\mathbb{R}^2$ is homeomorphic to

(a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
(b) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$
(c) $\{(x, y, z, w) \in \mathbb{R}^4 : x^2+y^2+z^2+w^2 = 1\}$
(d) $\mathbb{R}$

Q.2 Attempt any ***Seven***. [14]

(a) Give an examples of a function f such that $f \in BV[a, b]$ but $f \notin AC[a, b]$.
(b) Discuss the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(0,0) = 0$ and $f(x, y) = \frac{xy}{x^2+y^2}$ if $(x, y) \neq (0,0)$, at the point $(0,0)$.
(c) Show that $\mathbb{R}$ with upper limit topology is totally disconnected.
(d) Let X and Y be topological spaces. Show that the projection $\pi_1 : X \times Y \to X$, $\pi_1(x, y) = x$, is an open map.
(e) Prove or disprove: $\text{Log } i^2 = 2 \text{ Log } i$.
(f) Solve: $\sinh z = i$.

(P.T.O.)

(g) Prove: $|\sin x| \leq |\sin z|$ where $z = x + iy$.
(h) Determine last three digits of 19^{102}. State results which you use.
(i) List all abelian groups (upto isomorphism) of order 8.

Q.3

(a) Let f be an entire function satisfying $|f(z)| \leq M|z|^{\alpha}$, $(z \in \mathbb{C})$ for some $M > 0$ and $\alpha > 0$. Then show that f is a polynomial of degree $n \leq \alpha$. [6]
(b) Find minimum and maximum values of $|z^2 + 3z - 1|$ on $|z| \leq 1$. [6]

OR

(b) Find poles and zeros of $\frac{\tan z}{z^2}$. [6]

Q.4

(c) If order of an abelian group is divisible by 10, then show that it has a cyclic subgroup of order 10. [6]
(d) Does there exist a field of order 27? Justify. [6]

OR

(d) List all group homomorphisms from $\mathbb{Z}_6$ to S_3. [6]

Q.5

(e) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that $\lim_{n\to\infty} \int_a^b f(x) \sin nx dx = 0$. [6]
(f) Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that $f(x^+)$ and $f(x^-)$ exist for all $x \in \mathbb{R}$. Also show that the set of discontinuity of f is countable. [6]

OR

(f) Maximize the product $x_1x_2x_3$, where $x_i > 0$ and $x_1 + x_2 + x_3 = 3$. Deduce that $(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3}$ if $a, b, c > 0$. [6]

Q.6

(g) Let d be a metric on a set X. Let $\rho = \frac{d}{1+d}$. Prove the following statements. [6]
   (i) The metrics d and ρ generate the same topology on X.
   (ii) A sequence is Cauchy in (X, d) if and only if it is Cauchy in (X, ρ).
(h) Let $\mathbb{R}^{\omega}$ be the countable infinite product of $\mathbb{R}$ with itself. Show that $\mathbb{R}^{\omega}$ is not first countable with respect to the box topology. State the results you use. [6]

OR

(h) Define order topology on an ordered set X. Let X be a topological space, and let Y be an ordered set with the order topology. If $f, g : X \to Y$ are continuous, then show that the set $\{x \in X : f(x) \leq g(x)\}$ is closed in X. Deduce that the functions $\max\{f, g\}$ and $\min\{f, g\}$ are continuous. State the results you use. [6]

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[20] SEAT No.________

NO. OF PRINTED PAGES: 2

SARDAR PATEL UNIVERSITY

M.Sc. (Semester-IV) Examination
November 2017
Monday, 06 November, 2017
Time: 10:00 AM to 01:00 PM
Subject: Mathematics
Course No.PS04EMTH15 (Relativity-II)

Note: (1) All questions (including multiple choice questions) are to be answered in the answer book only.
(2) Numbers to the right indicate full marks of the respective question.
(3) Use the expressions in the Appendix if necessary.

Q-1 Choose most appropriate answer from the options given. (08)

(1) The metric tensor is a tensor of type ________.
(a) (1,1) (b) (2,0) (c) (0,2) (d) (0,0)

(2) Which one of the following is true?
(a) Christoffel symbols form a tensor (b) $\Gamma_{ii}^{h} = 0$
(c) $\Gamma_{ij}^{h} = \Gamma_{ji}^{h}$ (d) $\Gamma_{ih}^{h} = 0$

(3) A Riemannian space is empty if and only if ________
(a) $R_{hijk} = 0$ (b) $R_{ij} = 0$ (c) $R_{ij} \neq 0$ (d) $\Gamma_{ij}^{h} = 0$

(4) In Schwarzschild metric $r = 2m$ ________
(a) is a singularity (b) is a regular point
(c) always lies inside the body (d) is not a singularity

(5) Bending of light ________
(a) can not be explained using general relativity
(b) can not be explained using Newtonian theory
(c) happens only in Schwarzschild solution
(d) none of above is true

(6) Which one of the following is correct?
(a) de-Sitter universe is an Einstein space
(b) de-Sitter universe is not an Einstein space
(c) For de-Sitter universe $R_{ij} = 0$
(d) None of the above

(7) In Robertson-Walker model ________
(a) universe is ever expanding
(b) universe is static
(c) red-shift does not occur
(d) none of the above

(8) Which one of the following is non-static model?
(a) de-Sitter model
(b) Friedmann model
(c) Schwarzschild model
(d) none of above

Q-2 Answer any **Seven.** (14)

(1) Show that gradient of a scalar forms a tensor.
(2) State the formula for Christoffel symbols of second kind.
(3) State Ricci identity.
(4) What is meant by gravitational redshift?
(5) State Schwarzschild interior metric.

(6) What is the condition for a static model to be flat?
(7) State the expression of energy-momentum tensor for perfect fluid.
(8) What is the condition for the radiation filled universe?
(9) What is meant by Friedman models?

Q-3

(a) Define Christoffel symbols. Show that Christoffel symbols of second kind are symmetric in the lower indices. (06)

(b) Obtain geodesic equations for the space described by the metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. (06)

OR

(b) State the expression for the Riemann curvature tensor R_{hijk} and the Ricci tensor R_{ij}. Show that $R_{ij} = R_{ji}$.

Q-4

(a) Giving outline, derive Schwarzschild exterior solution. (06)

(b) State Schwarzschild exterior metric. Obtain its form in Kruskal coordinates. (06)

OR

(b) What are crucial tests of general relativity?

Q-5

(a) Obtain Einstein's field equations for a spherically symmetric static spacetime assuming the material content to be a perfect fluid. (06)

(b) State the metric for Einstein universe. Does Doppler effect occur in this model? Justify your answer. (06)

OR

(b) Prove or disprove, "de-Sitter universe is an Einstein space".

Q-6

(a) Discuss importance of cosmological principle and Weyl hypothesis. (06)

(b) Derive Robertson-Walker metric. (06)

OR

(b) For the Friedman models, obtain the expression for the age of the universe.

Appendix

For the spherically symmetric static metric $ds^2 = -e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu(r)}dt^2$, non-zero independent Christoffel symbols and components of Ricci tensor are as under,

$$\Gamma^1_{11} = \frac{\lambda'}{2},\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta,\ \Gamma^1_{44} = e^{\nu-\lambda}\frac{\nu'}{2},\ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{23} = \cot\theta,\ \Gamma^4_{14} = \frac{\nu'}{2}$$

$$R_{11} = \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} - \frac{\lambda'}{r},\ R_{22} = -1 + e^{-\lambda} - \frac{1}{2}re^{-\lambda}(\lambda' - \nu'),$$

$$R_{33} = \sin^2\theta\, R_{22},\quad R_{44} = -e^{\nu-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu'}{r}\right]$$

For R-W metric the non-zero components of Ricci tensor are given by,

$$\frac{R_{11}}{g_{11}} = \frac{R_{22}}{g_{22}} = \frac{R_{33}}{g_{33}} = \frac{\ddot{R}}{R} + 2\left(\frac{\dot{R}^2+k}{R^2}\right),\quad \frac{R_{44}}{g_{44}} = \frac{3\ddot{R}}{R}$$

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[62/A22] SEAT No.________

No. of printed pages: 2

SARDAR PATEL UNIVERSITY
M. Sc. (Semester IV) Examination

Date: 08-11-2017, Wednesday **Time: 10.00 To 01.00 p.m.**

Subject: MATHEMATICS **Paper No. PS04EMTH29 – (Graph Theory – II)**

Total Marks: 70

1. Choose the correct option for each question: [8]

(1) A shortest path between two vertices in a graph can be obtained using
(a) Kruscal algorithm (b) BFS algorithm
(c) Dijkstra's algorithm (d) none of these

(2) If all the digits in the Pruffer code are same, then the graph is
(a) Cycle graph (b) Path graph (c) Star graph (d) $K_{n,n}$ ($n > 1$)

(3) The number of spanning trees in C_n is
(a) n (b) n! (c) 1 (d) none of these

(4) In a network, if s is source and t is sink, then
(a) $d^+(s) = 0 = d^-(t)$ (b) $d^+(s) > 0, d^-(t) > 0$
(c) $d^+(s) = 0, d^-(t) > 0$ (d) $d^+(s) > 0, d^-(t) = 0$

(5) Let A be a matrix with spectrum $\{-1, -2, 2, 3\}$. Then Trace(A) =
(a) 12 (b) – 12 (c) –2 (d) 2

(6) Let G be a graph with $\chi(G) = 5$. Then $\lambda_{max}(G)$
(a) $= 4$ (b) ≤ 4 (c) ≥ 4 (d) none of these

(7) The Ramsey number R(3, 3) is
(a) 3 (b) 6 (c) 9 (d) none of these

(8) If $E = \{a, b, c\}$ with $M = \{\{a\}, \{b\}, \{a,b\}\}$ as hereditary system, then C_M =
(a) $\{c\}$ (b) $\{\{c\}, \{b,c\}\}$ (c) $\{\{c\}, \{a,c\}\}$ (d) $\{\{c\}, \{a,b,c\}\}$

2. Attempt any SEVEN: [14]

(a) How many trees are there with degree sequence (2,1,1,2,2)?

(b) State Matrix-tree theorem.

(c) If f is a flow on a network N = (V, A), then find $f(\{s\}, V)$ and $f(\{t\}, V)$.

(d) Prove or disprove: Length of minimum weighted path from vertex u to v is d(u, v).

(e) Prove: If G is k regular graph, then k is an eigen value of G.

(f) Prove: $\lambda_{max}(G) \leq \Delta(G)$.

(g) Define u-v separating set and give one example of it.

(h) Prove: $R(p, 2) = p$, if $p \geq 2$.

(i) Prove: For $X \subset E$ and $e \in E$, $r(X + e) \leq r(X) + 1$.

(PTO)

3. (a) Find $\tau(G)$ for $G = K_{2,3}$. [6]

(b) Construct a tree with Pruffer code (12321). [6]

OR

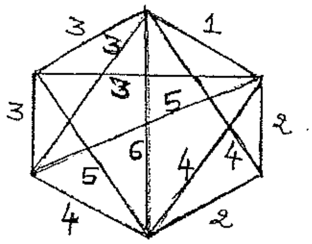
(b) Show that if a tree T with m edges has graceful labeling, then K_{2m+1} can be decomposed into (2m + 1) copies of T. [6]

4. (a) Let f be a flow on a network N = (V, A) with value d. Prove that, if $A(X, \overline{X})$ is a cut in N, then $d = f(X, \overline{X}) - f(\overline{X}, X)$. [6]

(b) Define a flow and value of a flow in a network and give one example of a flow in a network. [6]

OR

(b) Using Kruscal's algorithm, find a shortest spanning tree for the graph below: [6]



5. (a) Find $sp(C_4)$. [6]

(b) Let G be a bipartite graph. Prove that if λ is an eigenvalue of G with multiplicity m, then $-\lambda$ is also an eigenvalue of G with m multiplicity. [6]

OR

(b) Prove: The diameter of G is less than the number of distinct eigen values of G. [6]

6. (a) Prove: $R(p, q) \geq (p-1)(q-1) + 1$. [6]

(b) Prove (ANY ONE): In a hereditary system, [6]

(i) Sub modularity property (R) $\Rightarrow$ Weak elimination property (C).

(ii) Uniformity property (U) $\Rightarrow$ Base Exchange property (B).

x-x-x-x-x-x